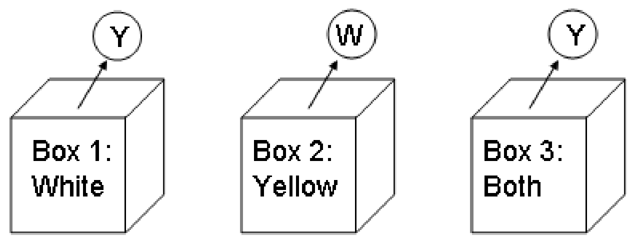
Example of Proofs by Natural Deduction and by Resolution Refutation

You are the proprietor of Sammy's Sport Shop. You have just received a shipment of three boxes filled with tennis balls. One box contains only yellow tennis balls, one box contains only white tennis balls, and one contains both yellow and white tennis balls. You would like to stock the tennis balls in appropriate places on your shelves. Unfortunately, the boxes have been labeled incorrectly; the manufacturer tells you that you have exactly one box of each, but that each box is definitely labeled wrong. One ball is drawn from each box and observed (assumed to be correct).



Given the initial (incorrect) labeling of the boxes above, and the three observations, use Propositional Logic to derive the correct labeling of the middle box.

- We can use propositional symbols like "O1Y" to mean a yellow ball was observed in box 1, "L1W" that the box was labeled white, and "C1B" to mean that it actually contains both, and so on.
- Do it in a general and complete way (such as what observing a white ball drawn from box 2 implies, that at most one box can contain yellow balls, etc.).
- Do not include derived knowledge that depends on the particular labeling of this instance shown above. Think of this knowledge base as a 'basis set' that could be used make inferences from any way the boxes could be labeled.
- Use propositional symbols in the following form: O1Y means a yellow ball was drawn (observed) from box 1, L1W means box 1 was initially labeled white, and C1B means box 1 actually contains both types of tennis balls.
- Finally, add the facts describing this particular situation to the knowledge base: {O1Y, O2W, O3Y, L1W, L2Y, L3B}

// what various observations imply
O1Y→C1Y∨C1B (c), O1W→C1W∨C1B
O2Y→C2Y∨C2B, O2W→C2W∨C2B (d)
O3Y→C3Y∨C3B, O3W→C3W∨C3B (f)
// labels are wrong
L1Y→¬C1Y, L1W→¬C1W, L1B→¬C1B
L2Y→¬C2Y, L2W→¬C2W, L2B→¬C2B
L3Y→¬C3Y, L3W→¬C3W, L3B→¬C3B (a)
// there is at least 1 box of each color
C1Y∨C1W∨C1B, C2Y∨C2W∨C2B, C3Y∨C3W∨C3B
// no 2 boxes have the same contents

C1Y→¬C2Y∧¬C3Y, C1W→¬C2W∧¬C3W, C1B→¬C2B∧¬C3B (e)
C2Y→¬C1Y∧¬C3Y, C2W→¬C1W∧¬C3W, C2B→¬C1B∧¬C3B
C3Y→¬C2Y∧¬C1Y (b), C3W→¬C2W∧¬C1W, C3B→¬C2B∧¬C1B

b) Use Natural Deduction to prove that box 2 contains white tennis balls (i.e. generate the sentence C2W using rules of inference).
proof of KB |= C2W by natural deduction:
1. from O3Y and (f) derive C3YvC3B by MP
2. from L3B and rule (a) above derive ~C3B by MP
3. from 1 and 3 derive C3Y by reso.
4. from 3 and (b) derive ~C1Y∧~C2Y by MP
5. from O1Y and (c) derive C1Y v C1B by MP
6. from 4 derive ~C1Y by AndElim
7. from 5 and 6 derive C1B by resol.
8. from O2W and (d) derive C2W v C2B by MP
9. from 7 and (e) derive ~C2B ∧~C3B
10. from 9 derive ~C2B
11. from 8 and 10 derive C2W by reso. // proof terminates with the query

c) Show that box 2 must contain white balls via a Resolution Refutation proof (*requires converting the sentences to CNF*).
// just the sentences I need...
O3Y→C3Y∨C3B (f) => 0. ¬O3Y∨C3Y∨C3B
O1Y→C1Y∨C1B (c) => 1. ¬O1Y∨C1Y∨C1B
O2W→C2W∨C2B (d) => 2. ¬O2W∨C2W∨C2B
L3B→¬C3B (a) => 3. ¬L3B∨¬C3B
C1B→¬C2B∧¬C3B (e) => 4a. ¬C1B∨¬C2B, 4b. ¬C1B∨¬C3B
C3Y→¬C2Y∧¬C1Y (b) => 5a. ¬C3Y∨¬C2Y, 5b. ¬C3Y∨¬C1Y
6. O1Y, // facts become unit clauses
7. O2W,
8. O3Y,
9. L1W,
10. L2Y,
11. L3B,
12. ¬C2W // negation of query
13. C3Y∨C3B [res, 8, 0]
14. ¬C3B [res, 11, 3]
15. C3Y [res, 13, 14]
16. ¬C1Y [res, 15, 5b]
17. C1Y∨C1B [res, 6, 1]
18. C1B [res, 16, 17]
19. ¬C2B [res, 18, 4a]
20. C2W∨C2B [res, 7, 2]
21. C2W [res, 19, 20]
22. □ [res, 21, 12] // proof terminates with the empty clause