CSCE 222
Discrete Structures for Computing

Sequences and Summations

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Based on slides by Andreas Klappenecker

Sequences

Sequences

A sequence is a function from a subset of the set of integers (such as {0,1,2,...} or {1,2,3,...}) to some set S.

We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

We use the notations $\{a_n\}$ or (a_n) to denote sequences.

Example

Let us consider the sequence $\{a_n\}$, where

$a_n = 1/n$.

Thus, the sequence starts with

$\{a_1, a_2, a_3, a_4, \ldots\} = \{1, 1/2, 1/3, 1/4, \ldots\}$

Sequences find ubiquitous use in computer science.

Geometric Progression

A geometric progression is a sequence of the form:

$a, ar, ar^2, ar^3, \ldots$

where the initial term a and the common ratio r are real numbers.

The geometric progression is a discrete analogue of an exponential function.

Arithmetic Progression

An arithmetic progression is a sequence of the form

a, $a+d$, $a+2d$, $a+3d$, ...

where the initial term a and the common difference d are real numbers.

An arithmetic function is a discrete analogue of a linear function $dx+a$.

Strings

The data type of a string is nothing but a sequence of finite length.

Recurrence Relations

Recurrence Relation

A recurrence relation for a sequence $\{a_n\}$ expresses the term a_n in terms of previous terms of the sequence.

The initial conditions for a recursively defined sequence specify the terms before the recurrence relation takes effect.

A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Example

Let $\{a_n\}$ be the sequence defined by the initial condition

$$a_0=2$$

and the recurrence relation

$$a_n = a_{n-1} + 3 \quad \text{for } n >= 1$$

Then $a_1 = 2+3=5$, $a_2=5+3=8$, $a_3=8+3=11, \ldots$

Fibonacci Sequence

Recall that the Fibonacci sequence is defined by the initial conditions: $f_0=0$ and $f_1=1$

and the recurrence relation

$$f_n = f_{n-1}+f_{n-2}$$

for $n >= 2$.

Hence, $\{f_n\} = \{0,1,1,2,3,5,8,13,...\}$

Solving Recurrence Relations

We say that we have solved a recurrence relation if we can find an explicit formula, called a closed formula, for the terms of the sequence.

Example: For the sequence given by the initial condition $a_0=2$ and the recurrence $a_n = a_{n-1} + 3$, we get the closed formula $a_n=3n+2$.

Fibonacci Sequence

Recall that the Fibonacci sequence is given by the initial conditions $f_0=0$ and $f_1=1$ and the recurrence relation $f_n = f_{n-1}+f_{n-2}$. A closed formula is given by

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Remark

There are many techniques available to solve recurrence relations. We will study some of them in depth later, including methods that allow us to derive the closed form solution to the Fibonacci sequence.

Summations

Sums

We use the notations

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

The letter *k* is called the index of summation.

Example

$$\begin{aligned}\sum_{k=1}^{5} k^2 &= 1^2 + 2^2 + 3^3 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55\end{aligned}$$

Remark

When counting the number of operations in the analysis of an algorithm, we get sums when counting the number of operations within a for loop.

The notation becomes particularly useful when counting operations in nested loops. The enumeration of terms with ellipses becomes tedious.

Geometric Series

If a and $r \neq 0$ are real numbers, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

Proof:

The case $r = 1$ holds, since $ar^j = a$ for each of the $n+1$ terms of the sum.

The case $r \neq 1$ holds, since

$$\begin{aligned} (r-1)\textstyle\sum_{j=0}^{n} ar^j &= \sum_{j=0}^{n} ar^{j+1} - \sum_{j=0}^{n} ar^j \\ &= \sum_{j=1}^{n+1} ar^j - \sum_{j=0}^{n} ar^j \\ &= ar^{n+1} - a \end{aligned}$$

and dividing by $(r-1)$ yields the claim.

Sum of the First *n* Positive Integers (1/2)

For all $n \geq 1$, we have

$$\sum_{k=1}^{n} k = n(n+1)/2$$

We prove this by induction.
Let $A(n)$ be the claimed equality.

Basis Step: We need to show that $A(1)$ holds.
For $n = 1$, we have

$$\sum_{k=1}^{1} k = 1 = 1(1+1)/2.$$

Sum of the First *n* Positive Integers (2/2)

Induction Step: We need to show that $\forall n \geq 1 : [A(n) \rightarrow A(n+1)]$
As induction hypothesis, suppose that $A(n)$ holds. Then,

$$
\begin{aligned}
\sum_{k=1}^{n+1} k &= (n+1) + \sum_{k=1}^{n} k \\
&= \frac{2(n+1)}{2} + \frac{n(n+1)}{2} \quad \text{by Induction Hypothesis} \\
&= \frac{2n+2+n^2+n}{2} = \frac{n^2+3n+2}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}
$$

Therefore, the claim follows by induction on n.

Sum of Fibonacci Numbers

Let $f_0 = 0$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Then

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1.$$

Induction basis: For $n = 1$, we have

$$\sum_{k=1}^{1} f_k = 1 = (1+1) - 1 = f_1 + f_2 - 1 = f_3 - 1$$

Sum of Fibonacci Numbers

Let $A(n)$ be the claimed equality.

Induction Step: We need to show that $\forall n \geq 1 : [A(n) \rightarrow A(n+1)]$
As induction hypothesis, suppose that $A(n)$ holds. Then,

$$\begin{aligned}
\sum_{k=1}^{n+1} f_k &= f_{n+1} + \sum_{k=1}^{n} f_k \\
&= f_{n+1} + f_{n+2} - 1 \quad \text{by Induction Hypothesis} \\
&= f_{n+3} - 1 \quad \text{by definition}
\end{aligned}$$

Therefore, the claim follows by induction on n.

Other Useful Sums

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Infinite Geometric Series

Let x be a real number such that $|x| < 1$. Then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Infinite Geometric Series

Since the sum of a geometric series satisfies

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1},$$

we have

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{x^{n+1}-1}{x-1}$$

As $\lim_{n\to\infty} x^{n+1} = 0$, we get

$$\sum_{k=0}^{\infty} x^k = \frac{-1}{x-1} = \frac{1}{1-x}.$$

Another Useful Sum

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Differentiating both sides of

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

yields the claim.