CSCE 222

Discrete Structures for Computing

Relations

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Based on slides by Andreas Klappenecker

Rabbits

Suppose we have three rabbits called Albert, Bertram, and Chris that have distinct heights.

Let us write (a,b) if a is taller than b.

Obviously, we cannot have both (Albert, Bertram) and (Bertram, Albert), so not all pairs of rabbit names will occur.

Suppose: Albert is taller than Bertram, and Bertram is taller than Chris.

Then the set of "taller than" relation is:

{ (Albert, Bertram), (Bertram, Chris), (Albert, Chris) }

Rabbits

Let

A = { Albert, Bertram, Chris }

be the set of rabbits.

Then the "taller than" relation is a subset of the cartesian product AxA, namely { (Albert, Bertram), (Bertram, Chris), (Albert, Chris) } $\subseteq$ A x A.

Binary Relations

Let A and B be sets.

A binary relation from A to B is a subset of AxB.

A relation on a set A is a subset of AxA.

Examples

Let us consider the following relations on the set of integers:

A = { (a,b) in **Z**x**Z** | a <= b }

B = { (a,b) in **Z**x**Z** | a > b }

C = { (a,b) in **Z**x**Z** | a = b or a = -b }

Notation

Let R be a relation from A to B. In other words, R contains pairs (a,b) with a in A and b in B.

If (a,b) in R, then we say that a is related to b by R.

It is customary to use infix notation for relations.

Thus, we write a R b to express that a is related to b by R. In other words, a R b if and only if (a,b) in R.

Example

Let A be the set of city names of the USA. Let B be the set of states. Define the relation C

C = { (a,b) in A x B | a is a city of b }

Then

(College Station, Texas)

(Austin, Texas)

(San Francisco, California)

all belong to the relation C.

Remark

The concept of a relation generalizes the concept of a function. A function f relates the argument x with its function value f(x). The difference is that a relation can relate an element x with more than one value.

For example, consider the relation

$$A = \{ (a,b) \text{ in } \mathbf{Z}\times\mathbf{Z} \mid a <= b \}.$$

Plan

We are going to study relations as mathematical objects. This allows us to abstract from well-known relations such as <=, =, "is taller than", "likes the same sport as".

We identify some basic properties of relations. Then we study relations generalizing the equality relation (so-called equivalence relations), and relations generalizing <= (so-called partial order relations).

Basic Properties of Relations

Reflexivity

We call a relation R on a set A reflexive if and only if $(a,a) \in R$ holds for all a in A.

Example: The equality relation = on the set of integers is reflexive, since $a=a$ holds for all integers a.

The less than relation $<$ on the set of integers is not reflexive, since $1<1$ does not hold.

Test Yourself...

X1 Let | denote the divides relation on the set of positive integers, so 2 | 4 means that there exists an integer x such that 2x=4. Is the relation | reflexive?

X2 Let S be the set of students in this class. Consider the relation R = "wears the same color shirt as." Is the relation R reflexive?

Symmetry

We call a relation R on a set A symmetric if and only if $(a,b) \in R$ implies that $(b,a) \in R$ holds.

Example: The equality relation = on the set of integers is symmetric, since $a=b$ implies that $b=a$.

The less than relation $<$ on the set of integers is not symmetric, since $1<2$ but $2<1$ does not hold.

Test Yourself...

X1 Let | denote the divides relation on the set of positive integers, so 2 | 4 means that there exists an integer x such that 2x=4. Is the relation | symmetric?

X2 Let S be the set of students in this class. Consider the relation R = "wears the same color shirt as". Is the relation R symmetric?

Antisymmetry

We call a relation R on a set A antisymmetric if and only if $(a,b) \in R$ and $(b,a) \in R$ imply that $a=b$.

Formally: $\forall a \forall b \; ((a,b) \in R \land (b,a) \in R) \longrightarrow a=b$.

Example: The equality relation = on the set of integers is antisymmetric, since $a=b$ and $b=a$ implies that $a=b$.

The less than relation < on the set of integers is antisymmetric. Why?

Test Yourself...

X1 Let | denote the divides relation on the set of positive integers, so 2 | 4 means that there exists an integer x such that 2x=4. Is the relation | antisymmetric?

X2 Let S be the set of students in this class. Consider the relation R = "wears the same color shirt as". Is the relation R antisymmetric?

Warning

The meaning of antisymmetry is not opposite to the meaning of symmetry! In fact, we have already seen that the equality relation = on the set of integers is both symmetric and antisymmetric.

You should very carefully study the meaning of these terms.

Transitive

We call a relation R on a set A transitive if and only if $(a,b) \in R$ and $(b,c) \in R$ imply that $(a,c) \in R$

Example: The equality relation = on the set of integers is transitive, since $a=b$ and $b=c$ implies that $a=c$.

The less than relation < on the set of integers is transitive, since $a<b$ and $b<c$ imply that $a<c$.

Test Yourself...

X1 Let | denote the divides relation on the set of positive integers, so 2 | 4 means that there exists an integer x such that 2x=4. Is the relation | transitive?

X2 Let S be the set of students in this class. Consider the relation R = "wears the same color shirt as". Is the relation R transitive?

Equivalence Relations

Equivalence Relation

A relation R on a set A is called an equivalence relation if and only if R is reflexive, symmetric, and transitive.

- Reflexive: For all a in A, we have $(a,a) \in R$

- Symmetric: (a,b) in R $\longrightarrow$ $(b,a) \in R$

- Transitive: $[\ (a,b) \in R$ and $(b,c) \in R\ ] \longrightarrow (a,c) \in R$

Example: Equality

The equality relation = on the set of integers is an equivalence relation.

Indeed,

the relation = is reflexive, since a=a holds for all integers a.

the relation = is symmetric, since a=b implies that b=a.

the relation = is transitive, since a=b and b=c implies that a=c.

Example: Congruence mod m

Let m be a positive integer. For integers a and b, we write

$$a \equiv b \pmod{m}$$

if and only if m divides a-b.

For all a in Z, we have m | (a-a), since m 0 = 0 = a-a. Thus, $a \equiv a \pmod{m}$ holds for all integers a. Thus, the relation is reflexive.

For a, b in Z, if $a \equiv b \pmod{m}$, then this means that there exists an integer k such that mk=a-b. Thus, m(-k) = b-a, which implies $b \equiv a \pmod{m}$. Thus, the relation is symmetric.

Example: Congruence mod m

If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ holds, then this means that there exist integers k and l such that

$mk = a-b$ and $ml = b-c$

Hence, $m(k+l) = a-b + b-c = a-c$

This shows that $a \equiv c \pmod{m}$ holds.

Therefore, the relation is transitive.

We can conclude that $a \equiv b \pmod{m}$ is an equivalence relation.

Equivalence Classes

Let R be an equivalence relation on a set A. For an element a in A, the set of elements

$$[a]_R = \{ b \text{ in } A \mid a \, R \, b \}$$

is called the equivalence class of a.

Example

Let us consider the equivalence relation $a \equiv b \pmod 4$ on the set of integers. Thus, two integers a and b are related whenever their difference is a multiple of 4. Thus, the equivalence classes are:

[0] = { ..., -8, -4, 0, 4, 8, ... }

[1] = { ..., -7,-3, 1, 5, 9, ... }

[2] = { ..., -6, -2, 2, 6,... }

[3] = { ..., -5, -1, 3, 7, ... }

Now note that [4] = [0], [5] = [1]. In fact, [0], [1], [2] and [3] are all equivalence classes.

Theorem

Let R be an equivalence relation on a set A. Then the following statements are equivalent:

a) $a \, R \, b$

b) $[a] = [b]$

c) $[a] \cap [b] \neq \emptyset$

Proof

Suppose that aRb holds. We are going to show that $[a] \subseteq [b]$ holds. Let $c \in [a]$. This means that aRc holds. Since R is symmetric, aRb implies that bRa. By transitivity, bRa and aRc imply that bRc holds. Hence, $c \in [b]$. Therefore, we have shown that $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar. Hence, we have shown that statement a) implies statement b).

We will show now that b) implies c). Since $a \in [a]$, we know that the equivalence class of a is not empty. As $[a] = [b] \neq \emptyset$, we have $[a] \cap [b] \neq \emptyset$.

Proof (continued)

We will show now that c) implies a).

Suppose that $[a] \cap [b] \neq \emptyset$. Thus, there exists an element c such that aRc and bRc. By symmetry, we get cRb. It follows by transitivity that aRb holds. $\square$

Partial Order Relations

Partial Orders

A relation R on a set A is called a partial order if and only if it is reflexive, antisymmetric, and transitive.

A set A with a partial order is called a partially ordered set (poset).

Example 1

The "less than or equal to" relation $\leq$ on the set of integers is a partial order relation.

Indeed, since $a \leq a$ holds for all integers a, the relation $\leq$ is reflexive.

Since $a \leq b$ and $b \leq a$ implies that $a = b$, the relation is antisymmetric.

Since $a \leq b$ and $b \leq c$ implies that $a \leq c$, the relation is transitive.

Example 2

The divides relation | on the set of positive integers is a partial order relation.

Indeed, since a|a for all positive integers a, the relation | is reflexive.

If a|b and b|a, then there exist integers k and l such that a k = b and b l = a. Therefore, a kl = a, so kl=1. This means that either k=l=1 or k=l=-1. Since a and b are positive integers, we cannot have a (-1) = b. Therefore, we must have k=l=1, which means that a=b. Thus, | is an antisymmetric relation.

The relation | is transitive, since a|b and b|c means that there exist integers k and l such that ak=b and bl=c, so a(kl)=c, which implies that a | c.

Test Yourself …

X1 Is the less than relation $<$ on the set of integers a partial order relation?

X2 Let S be a set. Is the subset relation $\subseteq$ on the set P(S) a partial order relation?

Comparable Elements

A partial order on a set S is often denoted by symbols resembling the notation commonly used for “less than or equal to”, namely $\leq$ or $\sqsubseteq$ or $\preccurlyeq$

Let (S, $\preccurlyeq$) be a partially ordered set. For two elements a and b of S, we do not necessarily have that one of the relations a $\preccurlyeq$ b or b $\preccurlyeq$ a holds. If one of them holds, then we call a and b comparable elements of S, otherwise a and b are incomparable.

Total Orders

A partially ordered set $(S, \preccurlyeq)$ in which any two elements are comparable is called a total order.

A totally ordered set is also called a chain.

For example, consider the set of positive integers N with <=. Any two positive integers are comparable with <=. It can form a chain such that 1 <= 2 <= 3 <= 4 <= ...

Lexicographic Ordering

Suppose that we have two partially ordered sets:

$(A, \preccurlyeq_1)$ and $(B, \preccurlyeq_2)$.

We can construct a partial order on AxB by defining

$(a_1, b_1) \preccurlyeq (a_2, b_2)$

if and only if $(a_1 = a_2$ and $b_1 \preccurlyeq_2 b_2)$ or $(a_1 \prec_1 a_2)$ holds.

We call the relation $\preccurlyeq$ the lexicographic order on the cartesian product AxB.

Example

Let **Z** be the set of integers, totally ordered with the "less than or equal to" relation $\leq$.

In the lexicographic order $\preccurlyeq$ on **Z**x**Z**, we have

$(3,4) \preccurlyeq (4,2)$

$(3,7) \preccurlyeq (3,8)$

Hasse Diagram

Let (S, <=) be a finite partially ordered set.

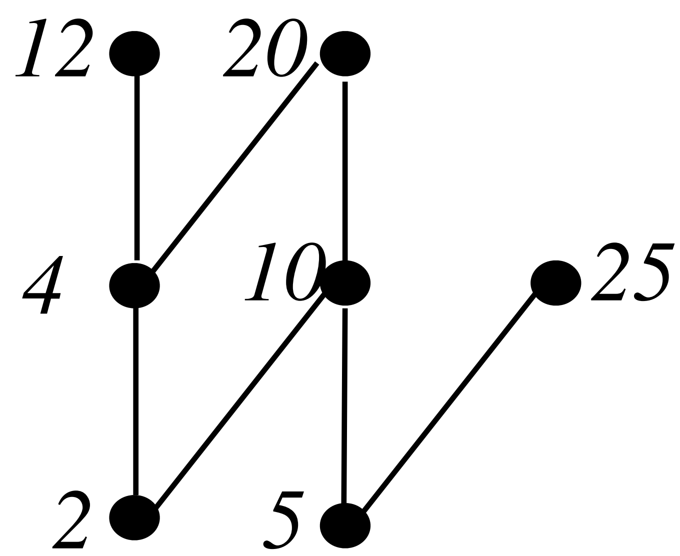
Suppose that a and b are distinct elements of S such that a <= b. We say that b covers a if and only if there does not exist an element c in S such that a < c < b.

The Hasse diagram of (S, <=) is a diagram in which an element b of S is written above a and connected by a line if and only if b covers a.

Examples

Consider {2,4,5,10,12,20,25} with divisibility condition.

The Hasse diagram is given by



The cover relation for this Hasse diagram is

{(2,4), (2,10), (4,12), (4,20), (5,10), (5,25), (10,20)}.

The Hasse diagram is more economical than representing the partial order relation by a directed graph (with an edge from a to b whenever a <= b). Self-loops and transitively implied relations are omitted.

Maximal and Minimal Elements

Let (S, <=) be a partially ordered set.

An element m in S is called maximal iff there does not exist any element b in S such that m < b.

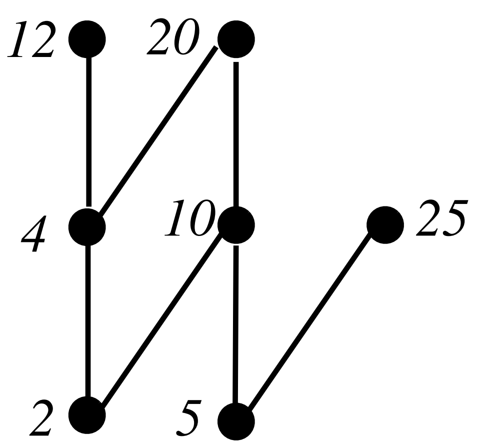
An element m in S is called minimal iff there does not exist any element b in S such that b < m.

Example

Determine the maximal elements of the set

{2,4,5,10,12,20,25},

partially ordered by the divisibility relation.



The elements 12, 20, and 25 are the maximal elements.

Determine the minimal elements of the above partially ordered set ({2,4,5,10,12,20,25}, |).

The elements 2 and 5 are the minimal elements.

Least and Greatest Element

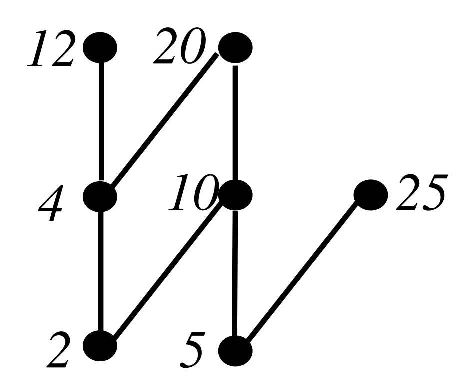
Let (S, <=) be a partially ordered set.

An element a in S is called the least element iff a <= b holds for all b in S.

[A least element does not need to exist. If it does, then it is uniquely determined.]

An element z in S is called the greatest element iff b <= z holds for all b in S.

[A greatest element does not need to exist. If it does, then it is uniquely determined.]



Test Yourself...

X1 Determine the least and the greatest elements of the set of positive integers partially ordered by divisibility.

X2 Let S be a nonempty set. Partially order the power set P(S) by inclusion. Determine the least and the greatest elements of P(S).

Lattices

Upper and Lower Bounds

Let (S, <=) be a partially ordered set.

Let A be a subset of S.

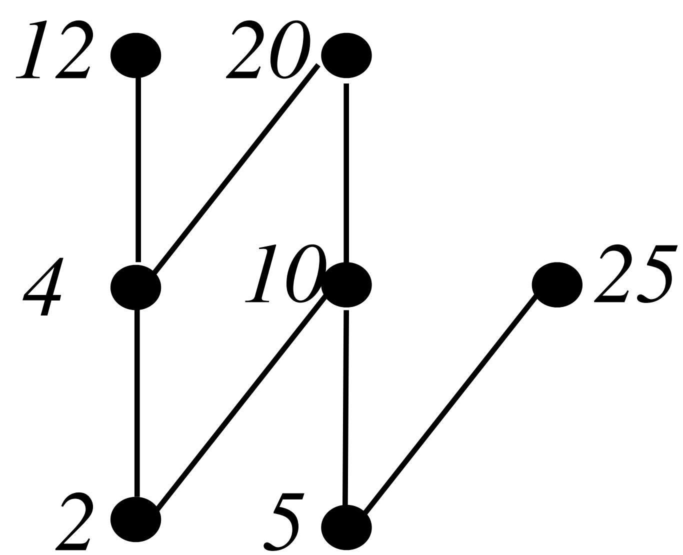
An element u of S is called an upper bound of A if and only if a <= u holds for all a in A.

An element l of S is called a lower bound of A if and only if l <= a holds for all a in A.

Example

Let {2,4,5,10,12,20,25} with divisibility condition.

The Hasse diagram is given by



The subset A = {4,10} has 20 as an upper bound, and 2 as a lower bound.

The subset A = {12} has 12 as an upper bound, and 2, 4 and 12 as lower bounds.

Least Upper Bounds

Let (S, <=) be a partially ordered set, and A a subset of S. An element u of S is called a least upper bound of A if it is an upper bound that is less than any other upper bound of A.

[Unlike upper bounds, the least upper bound is uniquely determined if it exists]

Greatest Lower Bounds

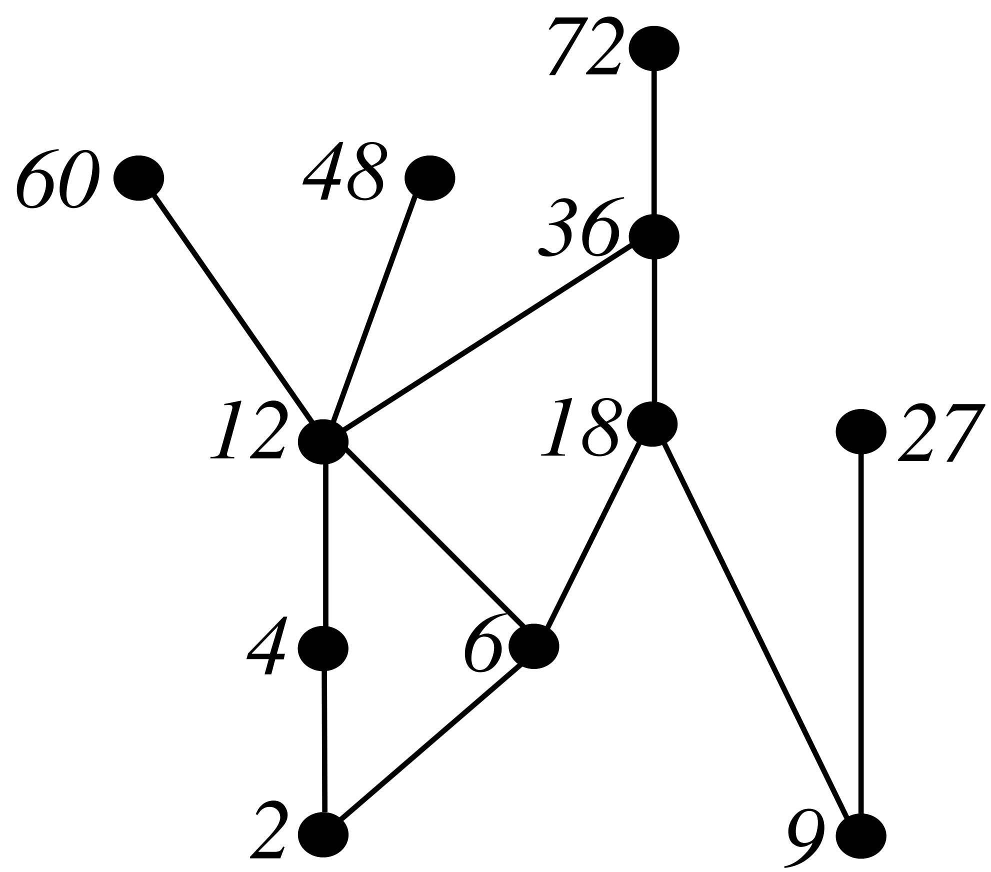
Let (S, <=) be a partially ordered set, and A a subset of S. An element l of S is called a greatest lower bound of A if it is a lower bound that is greater than any other lower bound of A.

A greatest lower bound is uniquely determined if it exists.

Example

Consider the poset (S={2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}, |).

Draw the Hasse diagram:



What are the upper bounds of the subset A={2, 9} ?

What are the lower bounds of the subset B={60, 72} ?

Lattices

A partially ordered set in which every pair has both a least upper bound and a greatest lower bound is called a lattice.

Example

Consider the set (N,|) of positive integers that is partially ordered with respect to the divisibility relation.

Let a and b be two distinct positive integers. Then the least upper bound of {a,b} is the least common multiple of a and b. The greatest lower bound is the greatest common divisor of {a,b}. Therefore, (N,|) is a lattice.