CSCE 222
Discrete Structures for Computing

Solving Recurrences

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Based on slides by Andreas Klappenecker

Motivation

We frequently have to solve recurrence relations in computer science.

For example, an interesting example of a heap data structure is a Fibonacci heap. This type of heap is organized with some trees. Its main feature are some lazy operations for maintaining the heap property. Analyzing the amortized cost for Fibonacci heaps involves solving the Fibonacci recurrence. We will outline a general approach to solve such recurrences.

The running time of divide-and-conquer algorithms requires solving some recurrence relations as well. We will review the most common method to estimate such running times.

Generating Functions

Given a sequence (a_0, a_1, a_2, a_3,...) of real numbers, one can form its generating function, an infinite series given by

$$\sum_{k=0}^{\infty} a_k x^k$$

The generating function is a formal power series, meaning that we treat it as an algebraic object, and we are not concerned with convergence questions of the power series.

Example 1

Suppose that the sequence is given by

$(k+1)_{k>=0}$

Then its generating function is given by

$$\sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + \cdots$$

Example 2

The generating function of the sequence $(1, 1, 1, 1, \ldots)$ is given by

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Example 3

The generating function of the sequence $(1, a, a^2, a^3, \ldots)$ is given by

$$1 + ax + a^2x^2 + a^3x^3 + \cdots = \frac{1}{1 - ax}.$$

Example 4

The generating function of $(1, 1, 1, 1, 1, 1, 0, 0, 0, \ldots)$ is given by

$$1 + x + x^2 + x^3 + x^4 + x^5.$$

$$\frac{x^6 - 1}{x - 1} = 1 + x + x^2 + x^3 + x^4 + x^5.$$

Sum of Sequences

Let $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ be sequences with generating functions $a(x)$ and $b(x)$, respectively.

Then $(a_k + b_k)_{k\geq 0}$ has the generating function

$$a(x) + b(x).$$

Products

$$\text{Let } a(x) = \sum_{k=0}^{\infty} a_k x^k \text{ and } b(x) = \sum_{k=0}^{\infty} b_k x^k.$$

Then

$$a(x)b(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

Example

Recall that the constant sequence $(1, 1, 1, 1, \ldots)$ has the generating function $1/(1-x)$.

What is the sequence corresponding to the generating function

$$\frac{1}{(1-x)^2} = \frac{1}{1-x}\frac{1}{1-x}?$$

$$\frac{1}{1-x}\frac{1}{1-x} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} 1\right) x^k = \sum_{k=0}^{\infty}(k+1)x^k$$

Table of Generating Functions

Study the table on page 568!

Solving Recurrences

Shifting Sequences

Let $G(x)$ be the generating function of the sequence $(a_k)_k$. Then

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Solving a Recurrence

Suppose we have the recurrence system with initial condition $a_0 = 2$ and recurrence $a_k = 3a_{k-1}$ for $k \geq 1$.

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1}x^k \\ &= a_0 + \sum_{k=1}^{\infty}(a_k - 3a_{k-1})x^k \\ &= 2. \end{aligned}$$

Thus, $G(x) - 3xG(x) = (1-3x)G(x) = 2$. Hence,

$$G(x) = \frac{2}{1-3x}.$$

Solving a Recurrence

Since we know that $1/(1-ax)=1+ax+a^2x^2+...$,

we have

$$G(x) = 2(1+3x+3^2x^2+...).$$

Therefore, a sequence solving the recurrence is given by

$$(2,2x3,2x3^2,...)=(2x3^k)_{k>=0}$$

Fibonacci Numbers (1)

The Fibonacci numbers satisfy the recurrence:

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \text{ for } n \geq 2 \end{aligned}$$

Fibonacci Numbers (2)

The Fibonacci numbers satisfy the recurrence:

$$
\begin{aligned}
f_0 &= 0 \\
f_1 &= 1 \\
f_2 &= f_1 + f_0 \\
f_3 &= f_2 + f_1 \\
f_4 &= f_3 + f_2 \\
&\vdots
\end{aligned}
$$

Fibonacci Numbers (3)

Let $F(x) = \sum_{k=0}^{\infty} f_k x^k$ be the generating function of the Fibonacci sequence. The recurrence

$$f_n = f_{n-1} + f_{n-2}$$

or

$$f_n - f_{n-1} - f_{n-2} = 0$$

suggests that we should consider

$$F(x) - xF(x) - x^2 F(x)$$

If it were not for the initial conditions, the latter sum would be identically 0.

Fibonacci Numbers (4)

The value of $F(x) - xF(x) - x^2F(x)$ is

$$\begin{array}{r} f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots \\ -(f_0x + f_1x^2 + f_2x^3 + \cdots) \\ -(f_0x^2 + f_1x^3 + \cdots) \\ \hline \end{array}$$

$$f_0 + (f_1 - f_0)x$$

Since $f_0 + (f_1 - f_0)x = x$, we get

$$F(x) - xF(x) - x^2F(x) = F(x)(1 - x - x^2) = x.$$

Hence,

$$F(x) = \frac{x}{1 - x - x^2}$$

Fibonacci Numbers (5)

We found the generating function

$$F(x) = \frac{x}{1 - x - x^2}$$

This function is of a simple form.

If we can find the coefficient of x^n in the power series of $F(x)$, we have found a closed form solution to the Fibonacci numbers.

Fibonacci Numbers (6)

Since $x/(1-x-x^2)$ is a rational function, we can use the method of partial fractions, which you might know from calculus.

Factor the denominator

$$1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x),$$

where

$$\alpha_1 = \frac{1+\sqrt{5}}{2}, \quad \alpha_2 = \frac{1-\sqrt{5}}{2}.$$

Then

$$\frac{x}{1-x-x^2} = \frac{A_1}{1-\alpha_1 x} + \frac{A_2}{1-\alpha_2 x}.$$

Fibonacci Numbers (7)

We can plug in various values for x to obtain linear equations that the real numbers A_1 and A_2 must satisfy. Solving these equations, we get

$$\frac{x}{1-x-x^2} = \frac{A_1}{1-\alpha_1 x} + \frac{A_2}{1-\alpha_2 x}$$

with

$$A_1 = \frac{1}{\sqrt{5}}, \qquad A_2 = -\frac{1}{\sqrt{5}}.$$

so

$$F(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha_1 x} - \frac{1}{1-\alpha_2 x}\right)$$

Fibonacci Numbers (8)

We can get the closed form by observing that

$$\begin{aligned} F(x) &= \tfrac{1}{\sqrt{5}}\left(\tfrac{1}{1-\alpha_1 x} - \tfrac{1}{1-\alpha_2 x}\right) \\ &= \tfrac{1}{\sqrt{5}}\left((1+\alpha_1 x + \alpha_1^2 x^2 + \cdots) - (1 + \alpha_2 x + \alpha_2^2 x^2 + \cdots)\right) \end{aligned}$$

Therefore,

$$f_n = \frac{1}{\sqrt{5}}(\alpha_1^n - \alpha_2^n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Recurrence Relations and Characteristic Polynomials

Linear Homogeneous Recurrence Relations

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

linear: a_n is a linear combination of a_k's
homogeneous: no terms occur that aren't multiples of a_k's
degree k: depends on previous k coefficients.

Example

$f_n = f_{n-1}+f_{n-2}$ is a linear homogeneous recurrence relation of degree 2.

$g_n = 5\ g_{n-5}$ is a linear homogeneous recurrence relation of degree 5.

$g_n = 5\ g_{n-5} + 2$ is a linear inhomogeneous recurrence relation.

$g_n = 5\ (g_{n-5})^2$ is a nonlinear recurrence relation.

Remark

Solving linear homogeneous recurrence relations can be done by generating functions, as we have seen in the example of Fibonacci numbers.

Now we will distill the essence of this method, and summarize the approach using a few theorems.

Fibonacci Numbers

Let $F(x)$ be the generating function of the Fibonacci numbers.

Expressing the recurrence $f_n = f_{n-1}+f_{n-2}$ in terms of $F(x)$ yields

$$F(x) = xF(x)+x^2F(x) + \text{corrections for initial conditions}$$

(the correction term for initial conditions is given by x).

We obtained: $F(x)(1-x-x^2) = x$ or $F(x) = x/(1-x-x^2)$

We factored $1-x-x^2$ in the form $(1-\alpha_1x)(1-\alpha_2x)$ and expressed the generating function $F(x)$ as a linear combination of

$$1/(1-\alpha_1x) \text{ and } 1/(1-\alpha_2x)$$

A Point of Confusion

Perhaps you might have been puzzled by the factorization

$$p(x) = 1-x-x^2 = (1-\alpha_1 x)(1-\alpha_2 x)$$

Writing the polynomial p(x) backwards (i.e. reciprocal polynomial),

$$c(x) = x^2 p(1/x) = x^2-x-1 = (x-\alpha_1)(x-\alpha_2)$$

yields a more familiar form. We call c(x) the characteristic polynomial of the recurrence $f_n = f_{n-1}+f_{n-2}$

Characteristic Polynomial

Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

be a linear homogeneous recurrence relation. The polynomial

$$x^k - c_1 x^{k-1} - \cdots - c_{k-1} x - c_k$$

is called the characteristic polynomial of the recurrence relation.

Remark: Note the signs!

Theorem

Let c_1, c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has two distinct roots r_1 and r_2. Then a sequence (a_n) is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for $n \geq 0$ for some constants α_1, α_2.

Example

Solve the recurrence system

$$a_n = a_{n-1} + 2a_{n-2}$$

with initial conditions $a_0 = 2$ and $a_1 = 7$.

The characteristic equation of the recurrence is

$$r^2 - r - 2 = 0.$$

The roots of this equation are $r_1 = 2$ and $r_2 = -1$. Hence, (a_n) is a solution of the recurrence iff

$$a_n = \beta_1 2^n + \beta_2 (-1)^n$$

for some constants β_1 and β_2. From the initial conditions, we get

$$\begin{array}{rcl} a_0 & = & 2 = \beta_1 + \beta_2 \\ a_1 & = & 7 = \beta_1 2 + \beta_2(-1) \end{array}$$

Solving these equations yields $\beta_1 = 3$ and $\beta_2 = -1$. Hence,

$$a_n = 3 \cdot 2^n - (-1)^n.$$

Further Reading

Our textbook discusses some more variations of the same idea. For example:

- How to solve recurrences which have characteristic equations with repeated roots (Theorem 2, page 544)

- How to solve recurrence of degree > 2

- How to solve recurrences of degree > 2 with repeated roots.

- How to solve certain inhomogeneous recurrences.

Divide-and-Conquer Algorithms and Recurrence Relations

Divide-and-Conquer

Suppose that you wrote a recursive algorithm that divides a problem of size n into

- a subproblems,

- each subproblem is of size n/b.

Additionally, a total of g(n) operations are required to combine the solutions.

How fast is your algorithm?

Divide-and-Conquer Recurrence

Let $f(n)$ denote the number of operations required to solve a problem of size n. Then

$$f(n) = a\, f(n/b) + g(n)$$

This is the divide-and-conquer recurrence relation.

Example: Binary Search

Suppose that you have a sorted array with n elements. You want to search for an element within the array. How many comparisons are needed?

Compare with median to find out whether you should search the left n/2 or the right n/2 elements of the array. Another comparison is needed to find out whether terms of the list remain.

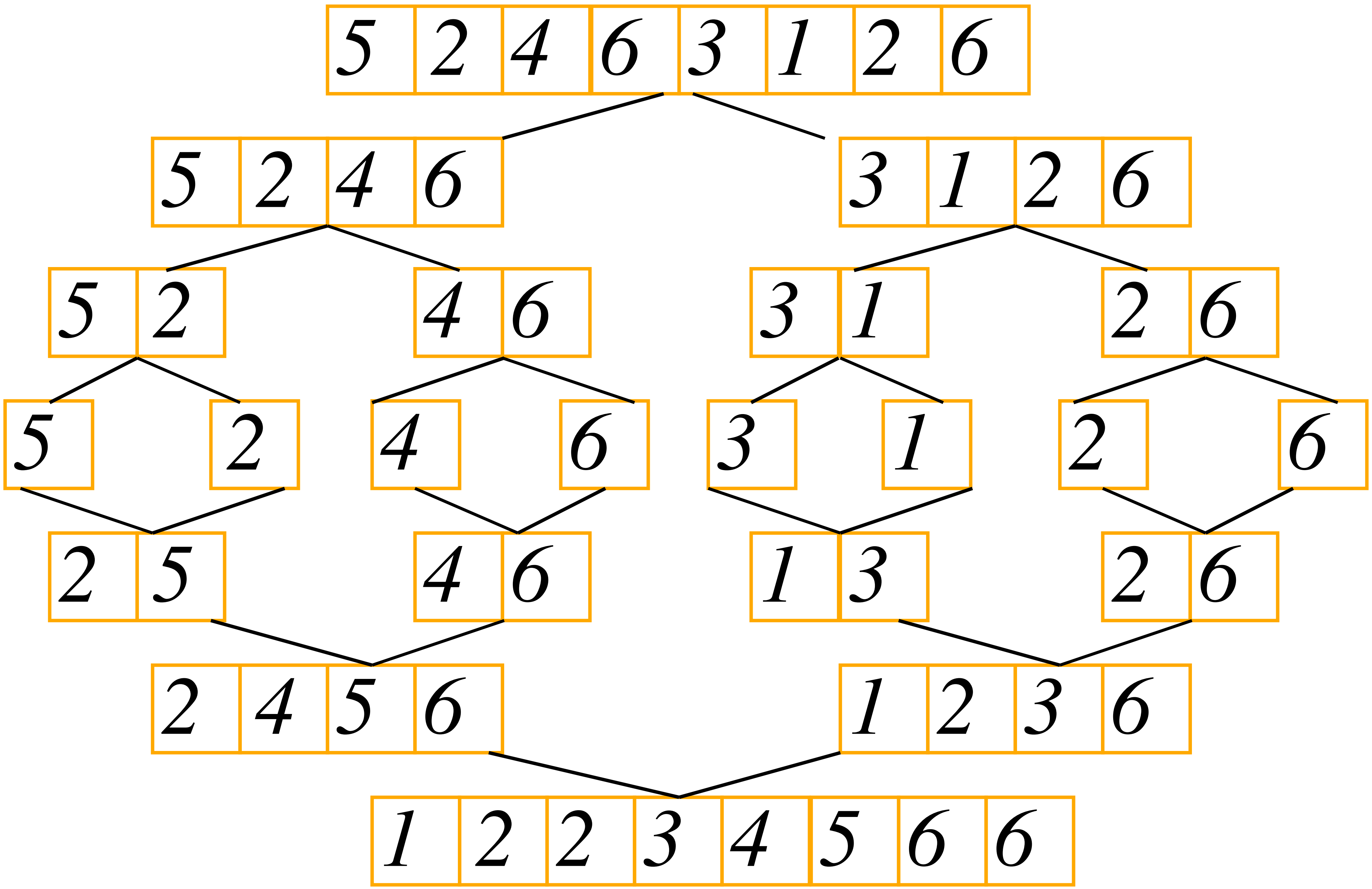
Thus, if f(n) is the number of comparisons, then

$$f(n) = f(n/2) + 2$$

Example: Mergesort

- DIVIDE the input sequence in half
- RECURSIVELY sort the two halves
  - basis of the recursion is sequence with 1 key
- COMBINE the two sorted subsequences by merging them

Mergesort Example



Recurrence Relation for Mergesort

- Let $T(n)$ be worst case time on a sequence of n keys
- If $n = 1$, then $T(n) = \Theta(1)$ (constant)
- If $n > 1$, then $T(n) = 2\,T(n/2) + \Theta(n)$
  - two subproblems of size $n/2$ each that are solved recursively
  - $\Theta(n)$ time to do the merge

Theorem

Let $f(n)$ be an increasing function satisfying the recurrence

$$f(n) = af(n/b) + c$$

whenever n is divisible by b, $a \geq 1$, $b > 1$ an integer, and c a positive real number. Then

$$f(x) = \begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1 \end{cases}$$

Proof

Suppose that $n = b^k$ for some positive integer k.

$$\begin{aligned}
f(n) &= af(n/b) + g(n) \\
&= a^2 f(n/b^2) + ag(n/b) + g(n) \\
&= a^3 f(n/b^3) + a^2 g(n/b^2) + ag(n/b) + g(n) \\
&\vdots \\
&= a^k f(n/b^k) + \textstyle\sum_{j=0}^{k-1} a^j g(n/b^j)
\end{aligned}$$

Suppose that $n = b^k$. For $g(n) = c$, we get

$$f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j c.$$

For $a = 1$, this yields

$$f(n) = f(1) + ck = f(1) + c\log_b n = O(\log n).$$

For $a > 1$, this yields

$$f(n) = a^k f(1) + c\frac{a^k - 1}{a - 1} = O(n^{\log_b a}).$$

If n is not a power of b, then estimate with the next power of b to get the claimed bounds.