CSCE 222
Discrete Structures for Computing

Recurrence Relations

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Based on slides by Andreas Klappenecker

Modeling with Recurrence Relations

Rabbits (1/3)

[From Leonardo Pisano's (a.k.a. Fibonacci) book Liber abaci]

A young pair of rabbits, one of each sex, is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month.

Assume that none of the rabbits die.

How many pair of rabbits are there after n months?

Rabbits (2/3)

Let f_n denote the number of pairs of rabbits after n months.

$f_1 = 1$ { reproducing pairs = 0, young pairs = 1 }

$f_2 = 1$ { reproducing pairs = 0, young pairs = 1 }

$f_3 = 2$ { reproducing pairs = 1, young pairs = 1 }

$f_4 = 3$ { reproducing pairs = 1, young pairs = 2 }

$f_5 = 5$ { reproducing pairs = 2, young pairs = 3 }

Rabbits (3/3)

The rabbit population can be modeled by a recurrence relation.

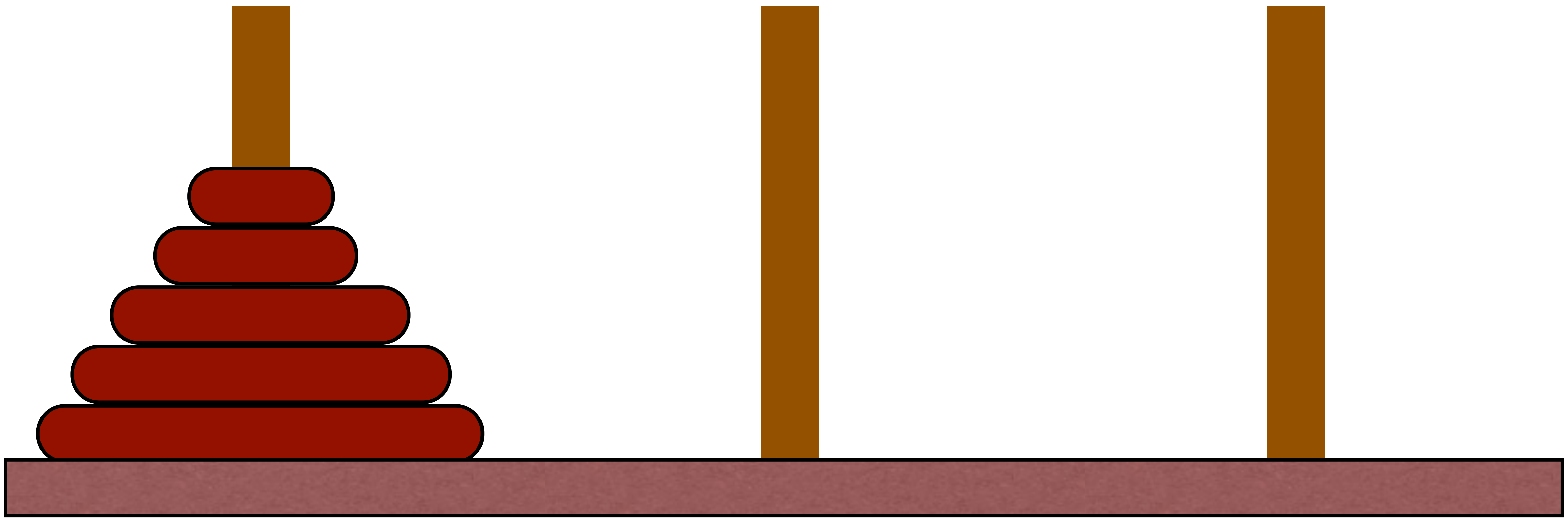
At the end of the first month, the number of pairs of rabbits on the island is $f_1 = 1$.

At the end of the second month, the number of pairs of rabbits on the island is $f_2 = 1$.

The number of pairs of rabbits after n months f_n is equal to the number of pairs of rabbits from the previous month f_{n-1} plus the number of pairs of newborn rabbits, which equals f_{n-2}, since each newborn pair comes from a pair that is at least two months old, so

$f_n = f_{n-1} + f_{n-2}$ for $n >= 3$.

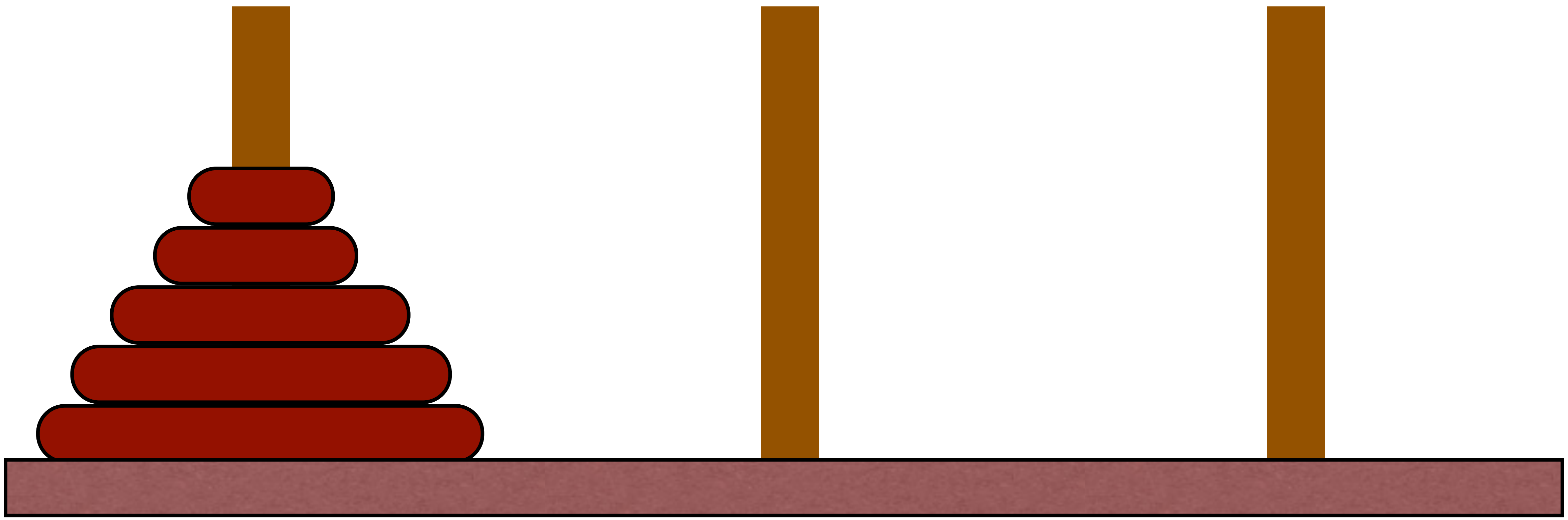
Tower of Hanoi



Initially, *n* discs are placed on the first peg. Move the *n* discs one at a time from one peg to another such that no larger disc is ever placed on a smaller disc.

Goal: Move the discs from peg 1 to peg 2.

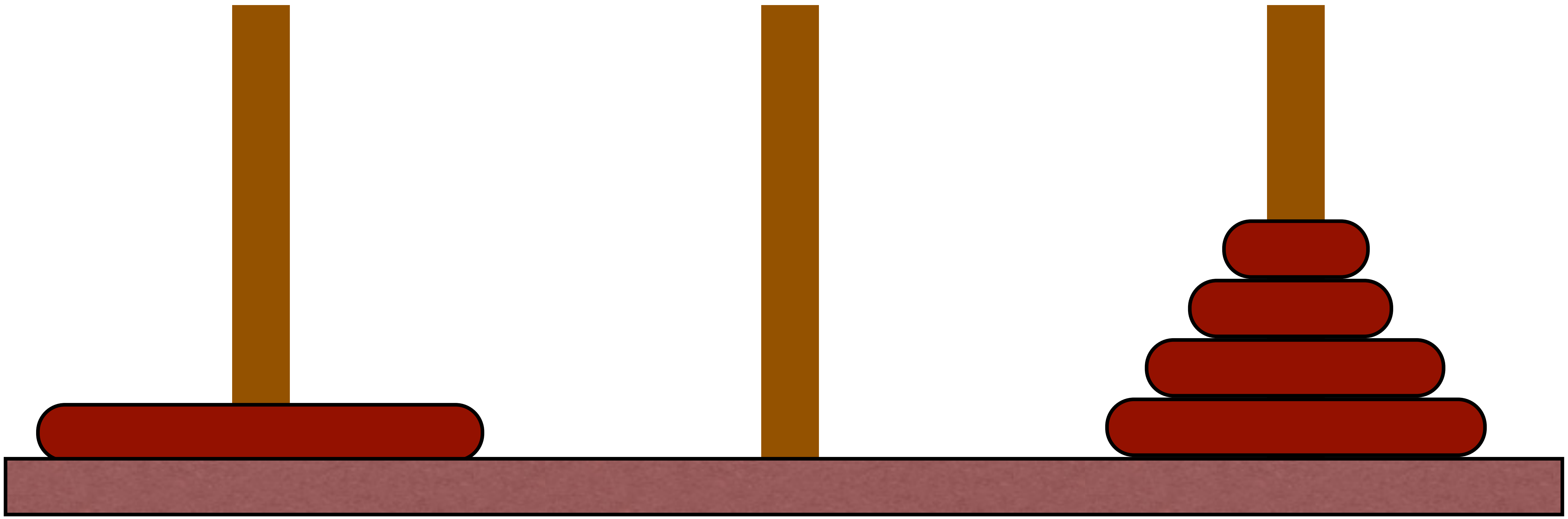
Tower of Hanoi



Let H_n denote the number of moves needed to solve the tower of Hanoi problem with *n* discs.

1) Move the top *n*-1 discs from peg 1 to peg 3 using H_{n-1} moves.

Tower of Hanoi

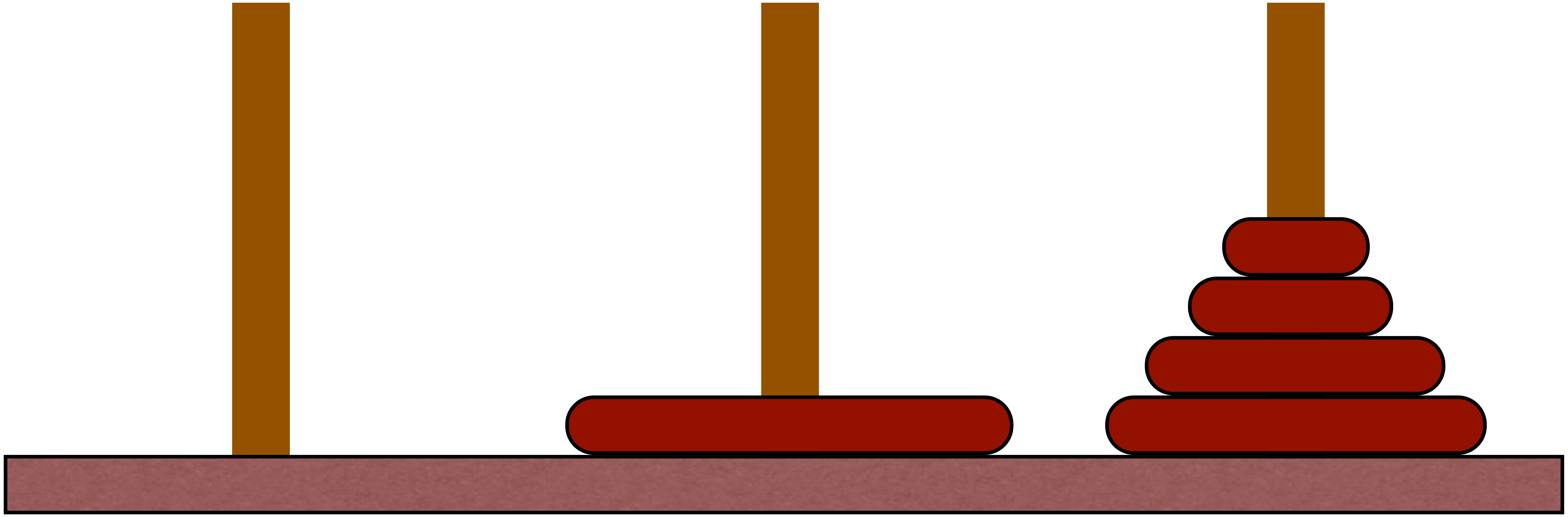


Let H_n denote the number of moves needed to solve the tower of Hanoi problem with n discs.

1) Move the top n-1 discs from peg 1 to peg 3 using H_{n-1} moves.

2) Move the largest disc from peg 1 to peg 2.

Tower of Hanoi



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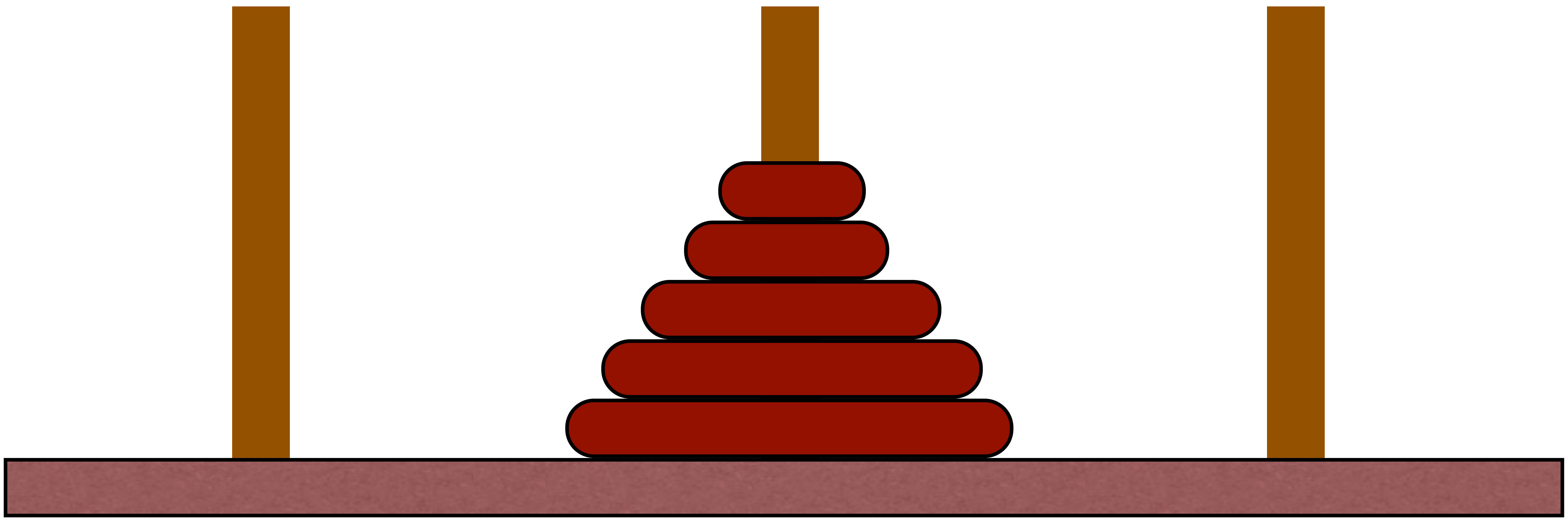
1) Move the top n-1 discs from peg 1 to peg 3 using H_{n-1} moves.

2) Move the largest disc from peg 1 to peg 2.

3) Move the n-1 discs from peg 3 to peg 2 using H_{n-1} moves.

Tower of Hanoi



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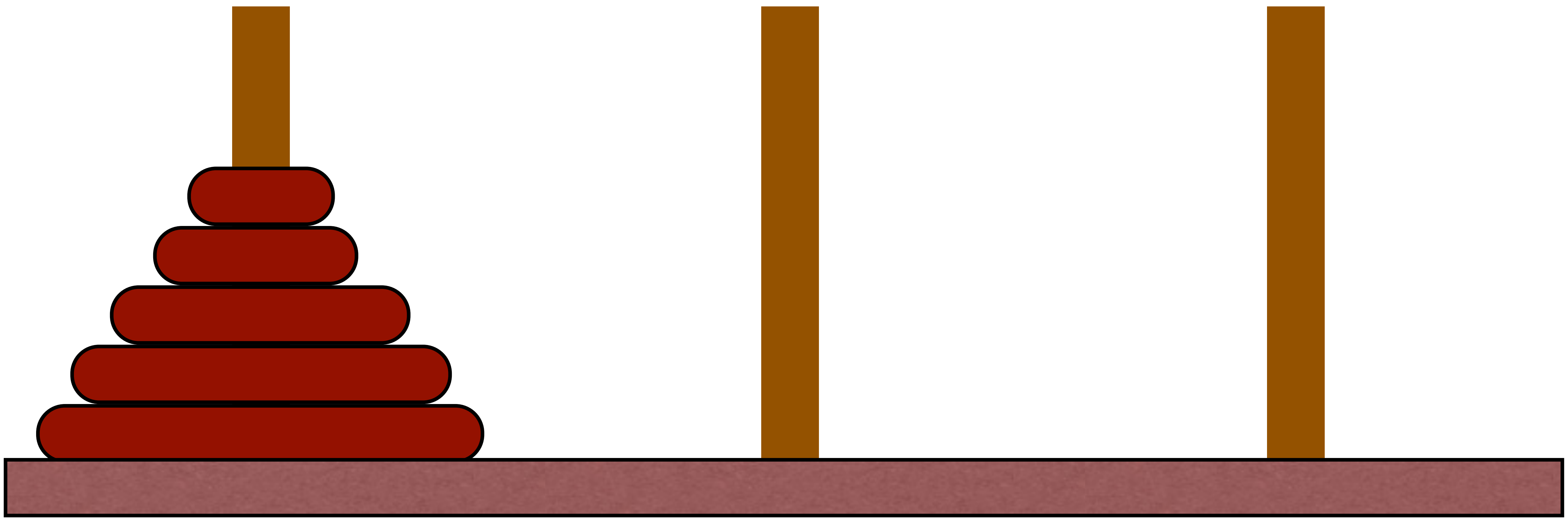
Tower of Hanoi



Let H_n denote the number of moves needed to solve the tower of Hanoi problem with n discs.

We have $H_n = H_{n-1} + 1 + H_{n-1} = 2H_{n-1} + 1$ for $n >= 2$,

and $H_1 = 1$.

Parenthesis

Let C_n denote the number of ways to parenthesize the product of n+1 numbers to specify the order of multiplication.

Example: We have $C_2 = 2$ since $(x_0\ x_1)\ x_2$ and $x_0\ (x_1\ x_2)$ are the only possibilities to parenthesize three numbers.

Example: We have $C_3 = 5$ since the product $x_0\ x_1\ x_2\ x_3$ can be parenthesized in the following five ways:

$((x_0\ x_1)\ x_2)\ x_3$ $\quad$ $(x_0\ (x_1\ x_2))\ x_3$

$(x_0\ x_1)(x_2\ x_3)$

$x_0\ ((x_1\ x_2)x_3)$ $\quad$ $x_0\ (x_1\ (x_2\ x_3))$

Parenthesis

Our goal is to find a recurrence relation for C_n.

1) Notice that one multiplication operator remains outside the parentheses, namely the one for the last multiplication to be performed.

2) If the last multiplication appears between x_k and x_{k+1} then there are $C_k C_{n-k-1}$ ways to set the remaining parentheses, as there are C_k ways to multiply the k+1 numbers x_0 x_1 ... x_k and the C_{n-k-1} ways to multiply the n-k numbers x_{k+1} ... x_n.

3) Since the final multiplication can appear between any of the n+1 numbers, we have $C_n = C_0\, C_{n-1} + C_1\, C_{n-2} + \ldots + C_{n-1}\, C_0$

4) We have $C_0=1$ and $C_1=1$.

Recurrence Relations

Recurrence Relation

A recurrence system is a finite set of initial conditions

$a_0 = C_0, a_1 = C_1, \ldots , a_k = C_k,$

where the $C_0, \ldots, C_k$ are real numbers, and a recurrence relation

$a_n = f(a_0, a_1, \ldots , a_{n-1})$

expressing a_n in terms of prior a_j with $j<n$.

A sequence $(a_0, a_1, \ldots)$ satisfying the initial condition and the recurrence relation is called a solution.

Example: Tower of Hanoi (1/2)

Recall that the number H_n of moves to solve the Tower of Hanoi puzzle satisfies the recurrence system:

Initial condition: $H_1 = 1$

Recurrence relation: $H_n = 2H_{n-1}+1$ for $n >= 2$.

For small values of n, we get

$(H_1, H_2, H_3, H_4,...) = (1,3,7,15,...)$

Therefore, we can guess that $H_n = 2^n - 1$

Example: Tower of Hanoi (2/2)

How can we prove that $H_n = 2^n - 1$ holds for all n>=1?

By induction.

Base Step: $H_1 = 1 = 2^1-1$, so our claim holds for n=1.

Inductive Step: As induction hypothesis (IH), suppose that $H_n = 2^n - 1$ holds. It follows that

$$H_{n+1} = 2H_n+1 \overset{\text{by IH}}{=} 2(2^n - 1) + 1 = 2^{n+1}-2+1=2^{n+1}-1$$

Therefore, the claim follows by induction on n.

Example: Fibonacci (1/3)

The recurrence system

Initial conditions: $f_0 = 0$, $f_1 = 1$

Recurrence: $f_n = f_{n-1} + f_{n-2}$ for $n >= 2$.

For small values of n, we get

$(f_0, f_1, f_2,...) = (0,1,1,2,3,5,8,13,...)$

We will later learn how to solve it. For any recurrence, we might try the encyclopedia of integer sequences

http://oeis.org

Example: Fibonacci (2/3)

$$f_n = \frac{1}{2^n\sqrt{5}}\left(\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right)$$

The formula might appear mysterious, since we have not yet learned how to derive it. Once we know about generating functions (or characteristic polynomials), it will be a routine matter to find this solution.

From this formula, it is not apparent why f_n should be an integer. So let's find out why this must be the case by expanding the formula.

Example: Fibonacci (3/3)

$$
\begin{aligned}
f_n &= \frac{1}{2^n\sqrt{5}}\left(\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right)\\
&= \frac{1}{2^n\sqrt{5}}\left(\sum_{k=0}^{n}\binom{n}{k}\sqrt{5}^k - \sum_{k=0}^{n}\binom{n}{k}(-\sqrt{5})^k\right)\\
&= \frac{1}{2^n\sqrt{5}}\sum_{\substack{k=0\\k\text{ odd}}}^{n} 2\binom{n}{k}\sqrt{5}^k\\
&= \frac{1}{2^{n-1}\sqrt{5}}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j+1}\sqrt{5}^{2j+1}\\
&= \frac{1}{2^{n-1}}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j+1}\sqrt{5}^{2j} = \frac{1}{2^{n-1}}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j+1}5^j
\end{aligned}
$$

Example: Parenthesis (1/3)

The number C_n of ways to parenthesize the product of n+1 numbers satisfies the recurrence system:

Initial conditions: $C_0=1$ and $C_1=1$

Recurrence relation: $C_n = C_0 C_{n-1} + C_1 C_{n-2} +...+ C_{n-1} C_0$

What is the solution to this recurrence system?

Example: Parenthesis (2/3)

The initial terms of the sequence are

$(C_0, C_1, C_2, C_3, \ldots) = (1,1,2,5,8,14,42,\ldots)$

Does this ring a bell?

No?

Check the online encyclopedia of integer sequences:

http://oeis.org (Catalan numbers)

Example: Parenthesis (3/3)

One can show that

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$