CSCE 222

Discrete Structures for Computing

Predicate Logic

Dr. Hyunyoung Lee

Based on slides by Andreas Klappenecker

Predicates

A function P from a set D to the set Prop of propositions is called a predicate.

The set D is called the domain of P.

Example

Let D=**Z** be the set of integers.

Let a predicate P: **Z** -> Prop be given by

P(x) = x>3.

The predicate itself is neither true or false. However, for any given integer the predicate evaluates to a truth value.

For example, P(4) is true and P(2) is false

Universal Quantifier (1)

Let P be a predicate with domain D.

The statement "P(x) holds for all x in D" can be written shortly as $\forall x P(x)$.

Universal Quantifier (2)

Suppose that $P(x)$ is a predicate over a finite domain, say $D=\{1,2,3\}$. Then

$\forall x P(x)$ is equivalent to $P(1) \wedge P(2) \wedge P(3)$.

Universal Quantifier (3)

Let P be a predicate with domain D.

$\forall x P(x)$ is true if and only if $P(x)$ is true for all x in D.

Put differently, $\forall x P(x)$ is false if and only if $P(x)$ is false for some x in D.

Existential Quantifier

The statement P(x) holds for some x in the domain D can be written as $\exists x\, P(x)$

Example: $\exists x\ (x>0 \land x^2 = 2)$

is true if the domain is the real numbers

but false if the domain is the rational numbers.

Logical Equivalence (1)

Two statements involving quantifiers and predicates are logically equivalent if and only if they have the same truth values no matter which predicates are substituted into these statements and which domain is used.

We write $A \equiv B$ for logically equivalent A and B.

Logical Equivalence (2)

$$\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$$

Proof: Suppose $\forall x(P(x) \wedge Q(x))$ is true. Then for all a in the domain $P(a) \wedge Q(a)$ is true. Hence, both $P(a)$ is true and $Q(a)$ is true. Since $P(a)$ is true for all a in the domain, $\forall x P(x)$ is true. Since $Q(a)$ is true for all a in the domain, $\forall x Q(x)$ is true. Hence $\forall x P(x) \wedge \forall x Q(x)$ is true.

[What else do we need to show?]

Logical Equivalence (3)

Suppose that $\forall x P(x) \wedge \forall x Q(x)$ is true.

It follows that $\forall x P(x)$ is true, and that $\forall x Q(x)$ is true. Hence, for each element a in the domain P(a) is true, and Q(a) is true. Hence $P(a) \wedge Q(a)$ is true for each element a in the domain.

Therefore, by definition, $\forall x(P(x) \wedge Q(x))$ is true.

De Morgan's Laws

The two versions of the de Morgan's laws for universal and existential quantifiers are given by

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

Example

$$\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x(P(x) \land \neg Q(x))$$

Proof:

$$\begin{aligned}
\neg\forall x(P(x) \rightarrow Q(x)) &\equiv \exists x \neg(P(x) \rightarrow Q(x)) \\
&\equiv \exists x \neg(\neg P(x) \lor Q(x)) \\
&\equiv \exists x(\neg\neg P(x) \land \neg Q(x)) \\
&\equiv \exists x(P(x) \land \neg Q(x))
\end{aligned}$$

Nested Quantifiers

Suppose that $P(x,y)$ is a predicate over a finite domain, say $Dx=\{1,2\}$ and $Dy=\{3,4\}$. Then

$\forall x \forall y P(x,y) \equiv [P(1,3) \wedge P(1,4)] \wedge [P(2,3) \wedge P(2,4)]$, and

$\exists x \forall y P(x,y) \equiv [P(1,3) \wedge P(1,4)] \vee [P(2,3) \wedge P(2,4)]$.

Nested Quantifiers Example (1)

"Each person's program is unique."

Domain is the set of persons.

$P(x,y)$: The program of person x is the same as that of person y.

$$\forall x \forall y((x \neq y) \rightarrow \neg P(x,y))$$

Nested Quantifiers Example (2)

$C(x,y)$: the program of person x crashes on input y

$L(x,y)$: the program of person x loops forever on input y

Domain for y is set of all inputs.

(1) $\exists y \forall x\, (C(x,y) \oplus L(x,y)) \not\equiv \forall x \exists y\, (C(x,y) \oplus L(x,y))$ (2)

(1) "There is an input that causes every person's program to crash or loop forever."

(2) "Every person's program has some input on which it either crashes or loops forever."

Quantified Predicate Example

The quantifiers can make definitions more memorable.

Recall that the limit

$$\lim_{x \to a} f(x) = L$$

is defined as: For every real number $\varepsilon>0$ there exists a $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever $0 < |x-a| < \delta$.

Quantified Predicate Example (2)

$$\lim_{x \to a} f(x) = L$$

For every real number $\varepsilon>0$ there exists a $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever $0 < |x-a| < \delta$.

$$\forall \varepsilon \exists \delta \forall x (0 < |x - a| < \delta \to |f(x) - L| < \varepsilon)$$

Quantified Predicate Example (3)

$$\lim_{x \to a} f(x) \overset{?}{=} L$$

What does it mean that the limit <u>does not</u> exist?

$$\neg \forall \varepsilon \exists \delta \forall x (0 < |x - a| < \delta \to |f(x) - L| < \varepsilon)$$
$$\equiv \exists \varepsilon \forall \delta \exists x \neg (0 < |x - a| < \delta \to |f(x) - L| < \varepsilon)$$
$$\equiv \exists \varepsilon \forall \delta \exists x \neg (\neg (0 < |x - a| < \delta) \vee |f(x) - L| < \varepsilon)$$
$$\equiv \exists \varepsilon \forall \delta \exists x (0 < |x - a| < \delta \wedge |f(x) - L| \geq \varepsilon)$$

Rules of Inference

Valid Arguments

An argument in propositional logic is a sequence of propositions that end with a proposition called conclusion. The argument is called valid if the conclusion follows from the preceding statements (called premises).

In other words, in a valid argument it is impossible that all premises are true but the conclusion is false.

Example

"If you have a current password, then you can log on to the network". (p->q)

"You have a current password" (p)

Therefore,

"You can log on to the network" (q)

Modus Ponens

The tautology $(p \land (p \rightarrow q)) \rightarrow q$ is the basis for the rule of inference called "modus ponens".

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore\ q \end{array}$$

Modus Tollens

$\neg q$

$p \rightarrow q$

$\therefore \neg p$

"The University will not close on Thursday."

"If it snows on Thursday, then the University will close."

Therefore, "It will not snow on Thursday"

Simplification

$p \wedge q$

$\therefore p$

See Table 1 Rules of Inference on page 76 for more rules.

Example Formal Argument

$$\begin{array}{l} \neg p \wedge q \\ r \rightarrow p \\ \neg r \rightarrow s \\ s \rightarrow t \\ \hline \\ \therefore t \end{array}$$

Argument

1) $\neg p \wedge q$	Hypothesis
2) $\neg p$	Simplification of 1)
3) $r \rightarrow p$	Hypothesis
4) $\neg r$	Modus tollens using 2) and 3)
5) $\neg r \rightarrow s$	Hypothesis
6) s	Modus ponens using 4) and 5)
7) $s \rightarrow t$	Hypothesis
8) t	Modus ponens using 6) and 7)

Formal Argument in Action

$$\begin{array}{l} \neg p \wedge q \\ r \rightarrow p \\ \neg r \rightarrow s \\ s \rightarrow t \\ \hline \therefore t \end{array}$$

p: "It will rain on Sunday"

q: "It will be warm on Sunday"

r: "It will be cloudy on Sunday"

s: "It will be sunny on Sunday"

t: "We will do picnic on Sunday"

$\neg p \wedge q$ "It will not rain and it will be warm."

$r \rightarrow p$ "If it is cloudy then it will rain."

$\neg r \rightarrow s$ "If it is not cloudy then it will be sunny."

$s \rightarrow t$ "If it is sunny, then we will do picnic."

Quantified Statements

$\dfrac{\forall x P(x)}{\therefore P(a)}$	*Universal instantiation*
$\dfrac{P(a) \text{ for an arbitrary } a}{\therefore \forall x P(x)}$	*Universal generalization*
$\dfrac{\exists x P(x)}{\therefore P(a) \text{ for some } a}$	*Existential instantiation*
$\dfrac{P(a) \text{ for some } a}{\therefore \exists x P(x)}$	*Existential generalization*

Universal Modus Ponens

Let us combine universal instantiation and modus ponens to get the "universal modus ponens" rule of inference.

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ P(a) \text{ where } a \text{ is in the domain} \\ \hline \therefore Q(a) \end{array}$$

For example, assume that "For all positive integers n, if n>4, then $n^2<2^n$" is true. Then the universal modus ponens implies that $100^2<2^{100}$.