CSCE 222
Discrete Structures for Computing

Proof by Induction

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Based on slides by Andreas Klappenecker

Motivation

Induction is an axiom which allows us to prove that certain properties are true for all positive integers (or for all nonnegative integers, or all integers >= some fixed number)

Induction Principle

Let $A(n)$ be an assertion concerning the integer n.

If we want to show that $A(n)$ holds for all positive integer n, we can proceed as follows:

Induction basis: Show that the assertion $A(1)$ holds.

Induction step: For all positive integers n, show that $A(n)$ implies $A(n+1)$.

Standard Example: Sum of the First *n* Positive Integers (1/2)

For all $n \geq 1$, we have

$$\sum_{k=1}^{n} k = n(n+1)/2$$

We prove this by induction.
Let $A(n)$ be the claimed equality.

Basis Step: We need to show that $A(1)$ holds.
For $n = 1$, we have

$$\sum_{k=1}^{1} k = 1 = 1(1+1)/2.$$

Sum of the First *n* Positive Integers (2/2)

Induction Step: We need to show that $\forall n \geq 1 : [A(n) \to A(n+1)]$.
As induction hypothesis, suppose that $A(n)$ holds. Then,

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) && \text{by definition of } \sum \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} && \text{by induction hypothesis} \\
&= \frac{(n+1)(n+2)}{2} && \text{by factoring out } (n+1)
\end{aligned}$$

Therefore, the claim follows by induction on n.

The Main Points

We established in the induction basis that the assertion $A(1)$ is true.

We showed in the induction step that $A(n+1)$ holds, assuming that $A(n)$ holds.

In other words, we showed in the induction step that $A(n) \longrightarrow A(n+1)$ holds for all $n >= 1$.

Perfect Squares

The perfect squares are given by

$1^2=1$, $2^2=4$, $3^2=9$, $4^2=16$, …

			□ □ □ □
		□ □ □	⧆ ⧆ ⧆ □
	□ □	⧆ ⧆ □	⧆ ⧆ ⧆ □
□	⧆ □	⧆ ⧆ □	⧆ ⧆ ⧆ □

$(n+1)^2 = n^2+n+n+1 = n^2+2n+1$

$1+3+5+7 = 4^2$

Example 2

Theorem: For all positive integers n, we have

$$1+3+5+...+(2n-1) = n^2$$

Proof. We prove this by induction on n. Let $A(n)$ be the assertion of the theorem.

Induction basis: Since $1 = 1^2$, it follows that $A(1)$ holds.

Induction step: As induction hypothesis (IH), suppose that $A(n)$ holds. Then

$$1+3+5+...+(2n-1)+(2n+1) \overset{\text{by IH}}{=} n^2+(2n+1) = (n+1)^2$$

holds. In other words, $A(n)$ implies $A(n+1)$.

Therefore, the claim follows by induction on n.

Example 3

Theorem: We have

$1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6$

for all $n >= 1$.

Proof. Your turn!!!

Let $B(n)$ denote the assertion of the theorem.

Induction basis:

Since $1^2 = 1(1+1)(2+1)/6$, we can conclude that $B(1)$ holds.

Example 3 (Cont.)

Inductive step: As induction hypothesis (IH), suppose that $B(n)$ holds. Then

$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2$ by IH

Factoring out $(n+1)$ on the right hand side yields

$(n+1)(n(2n+1)+6(n+1))/6 = (n+1)(2n^2+7n+6)/6$

One easily verifies that this is equal to

$(n+1)(n+2)(2n+3)/6 = (n+1)((n+1)+1)(2(n+1)+1)/6$

Thus, $B(n+1)$ holds.

Therefore, the claim follows by induction on n.

Example 4

Theorem: We have

$1^3 + 2^3 + ... + n^3 = n^2(n+1)^2/4$

for all $n >= 1$.

Proof. Let $P(n)$ denote the assertion of the theorem.

Induction basis: Show that $P(1)$ holds.

Since $1^3 = 1^2(1+1)^2/4$, we conclude that $P(1)$ holds.

Example 4 (Cont.)

Inductive step: As induction hypothesis (IH), suppose that $P(n)$ holds. Then

$1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = n^2(n+1)^2/4 + (n+1)^3$ by IH

Factoring out $(n+1)^2$ on the right hand side yields

$(n+1)^2(n^2+4(n+1))/4 = (n+1)^2(n^2+4n+4)/4 = (n+1)^2(n+2)^2/4$

which is equal to

$(n+1)^2((n+1)+1)^2/4$

Thus, $P(n+1)$ holds.

Therefore, the claim follows by induction on n.

Tip

How can you verify whether your algebra is correct?

Use http://www.wolframalpha.com

[Not allowed in any exams, though. Sorry!]

More Examples

Sum of Fibonacci Numbers (1/2)

Let $f_0 = 0$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Then

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1.$$

Induction basis: For $n = 1$, we have

$$\sum_{k=1}^{1} f_k = 1 = (1+1) - 1 = f_1 + f_2 - 1 = f_3 - 1$$

Sum of Fibonacci Numbers (2/2)

Let $A(n)$ be the claimed equality.

Induction Step: We need to show that $\forall n \geq 1 : [A(n) \rightarrow A(n+1)]$
As induction hypothesis, suppose that $A(n)$ holds. Then,

$$\begin{aligned}
\sum_{k=1}^{n+1} f_k &= f_{n+1} + \boxed{\sum_{k=1}^{n} f_k} \\
&= f_{n+1} + \boxed{f_{n+2} - 1} \quad \text{by } \boxed{\text{Induction Hypothesis}} \\
&= f_{n+3} - 1 \quad \text{by definition}
\end{aligned}$$

Therefore, the claim follows by induction on n.

Factorials

Theorem. $\sum_{i=0}^{n} i(i!) = (n+1)! - 1$. (By convention, $0! = 1$.)

Induction Basis: For $n = 0$.
Since $\sum_{I=0}^{0} i(i!) = 0(0!) = 0 = 1 - 1 = (0+1)! - 1$, the claim holds for $n = 0$.
Induction Step: As induction hypothesis (IH), suppose the claim is true for n.
Then,

$$\begin{aligned}\sum_{i=0}^{n+1} i(i!) &= \sum_{i=0}^{n} i(i!) + (n+1)(n+1)! \\ &= (n+1)! - 1 + (n+1)(n+1)! \quad \text{by IH} \\ &= (n+2)(n+1)! - 1 \quad \text{by factoring out } (n+1)! \\ &= (n+2)! - 1 \quad \text{by definition of } !\end{aligned}$$

Therefore, the claim follows by induction on n.

Divisibility

Theorem: For all positive integers *n*, the number

$7^n - 2^n$

is divisible by 5.

Proof: By induction.

Induction basis. Since 7-2=5, the theorem holds for $n=1$.

Divisibility

Inductive step:

Suppose that 7^n-2^n is divisible by 5. Our goal is to show that this implies that 7^{n+1}-2^{n+1} is divisible by 5. We note that

$7^{n+1}-2^{n+1} = 7x7^n-2x2^n= 5x7^n+2x7^n-2x2^n = 5x7^n +2(7^n-2^n)$.

By induction hypothesis, $(7^n-2^n) = 5k$ for some integer k.

Hence, $7^{n+1}-2^{n+1}= 5x7^n +2x5k = 5(7^n +2k)$, so

$7^{n+1}-2^{n+1} =5$ x some integer.

Thus, the claim follows by induction on n.

Strong Induction

Strong Induction

Suppose we wish to prove a certain assertion concerning positive integers.

Let $A(n)$ be the assertion concerning the integer n.

To prove it for all $n >= 1$, we can do the following:

1) Prove that the assertion $A(1)$ is true.

2) Assuming that the assertions $A(k)$ are proved for all $k<n$, prove that the assertion $A(n)$ is true.

We can conclude that $A(n)$ is true for all $n>=1$.

Strong Induction

Induction basis:

Show that $A(1)$ is true.

Induction step:

Show that $(\underbrace{A(1) \wedge A(2) \wedge \ldots \wedge A(n)}_{\text{strong induction } hypothesis}) \rightarrow A(n+1)$

holds for all $n >= 1$.

Postage

Theorem: Every amount of postage that is at least 12 cents can be made from 4¢ and 5¢ stamps.

Postage

Proof by induction on the amount of postage.

Induction Basis: If the postage is

12¢: use three 4¢ and zero 5¢ stamps (12=3x4+0x5)

13¢: use two 4¢ and one 5¢ stamps (13=2x4+1x5)

14¢: use one 4¢ and two 5¢ stamps (14=1x4+2x5)

15¢: use zero 4¢ and three 5¢ stamps (15=0x4+3x5)

(Not part of induction basis, but let us try some more)

16¢: use (three+one) 4¢ and zero 5¢ stamps ((3+1)x4+0x5)

17¢: use (two+one) 4¢ and one 5¢ stamps ((2+1)x4+1x5)

18¢: use (one+one) 4¢ and two 5¢t stamps ((1+1)x4+2x5)

19¢: use (zero+one) 4¢ and three 5¢ stamps ((0+1)x4+3x5)

20¢: use (three+two) 4¢ and zero 5¢ stamps ((3+2)x4+0x5)

. . .

Postage

Inductive step:
Suppose that we have shown how to construct postage for every value from 12 up through k. We need to show how to construct $k + 1$ cents of postage.

Since we've already proved the induction basis, we may assume that $k + 1 \geq 16$. Since $k+1 \geq 16$, we have $(k+1)-4 \geq 12$. By inductive hypothesis, we can construct postage for $(k + 1) - 4$ cents using m 4¢ stamps and n 5¢ stamps for some non-negative integers m and n. In other words $((k + 1) - 4) = 4m + 5n$; hence, $k+1 = 4(m+1)+5n$.

Therefore, the claim follows by strong induction on n.

Quiz

Why did we need to establish four cases in the induction basis?

Isn't it enough to remark that the postage for 12 cents is given by three 4 cents stamps?

Another Example: Sequence

Theorem: Let a sequence (a_n) be defined as follows:

$a_0=1$, $a_1=2$, $a_2=3$,

$a_k = a_{k-1}+a_{k-2}+a_{k-3}$ for all integers $k \geq 3$.

Then $a_n \leq 2^n$ for all integers $n \geq 0$. $P(n)$

Proof. Induction basis:

The statement is true for $n=0$, since $a_0=1 \leq 1=2^0$ $P(0)$

for $n=1$: since $a_1=2 \leq 2=2^1$ $P(1)$

for $n=2$: since $a_2=3 \leq 4=2^2$ $P(2)$

Sequence (cont'd)

Inductive step:

(S.I.H.) Assume that $P(i)$ is true for all i with $0 \le i < k$, that is, $a_i \le 2^i$ for all $0 \le i < k$, where $k > 2$.

Show that $P(k)$ is true: $a_k \le 2^k$

$$\begin{aligned} a_k &= a_{k-1} + a_{k-2} + a_{k-3} && \text{by def. of seq.} \\ &\le 2^{k-1} + 2^{k-2} + 2^{k-3} && \text{by S.I.H.} \\ &\le 2^0 + 2^1 + \ldots + 2^{k-3} + 2^{k-2} + 2^{k-1} \\ &= 2^k - 1 \le 2^k && \text{by understanding binary number system} \end{aligned}$$

$$\begin{array}{rrrrrr} 2^3 & 2^2 & 2^1 & 2^0 & & \\ \downarrow & \downarrow & \downarrow & \downarrow & & \\ 1 & 0 & 0 & 0_2 & = & 8_{10} \\ - & & & 1_2 & & -\ 1_{10} \\ \hline & 1 & 1 & 1_2 & = & 7_{10} \end{array}$$

Thus, $P(n)$ is shown true for all integers $n \ge 0$ by strong induction.

Yet Another Example Sequence

A sequence a_0, a_1, a_2, ... is defined recursively as follows:

$a_0 = 0$;

$a_1 = 1$;

$a_n = 5a_{n-1} - 6a_{n-2}$ for all $n >= 2$.

Prove that for all non-negative integers n, $a_n = 3^n - 2^n$: $P(n)$

Proof. Induction basis: need to show $P(0)$ and $P(1)$ hold.

$P(0)$ holds since $a_0 = 0 = 1 - 1 = 3^0 - 2^0$

$P(1)$ holds since $a_1 = 1 = 3 - 2 = 3^1 - 2^1$

Yet Another Example Sequence (Cont.)

Inductive step:

(S.I.H.) Assume that $P(i)$ is true for all i with $0 \leq i < n$, that is, $a_i = 3^i - 2^i$ for all $0 \leq i < n$, where $n > 1$.

Show that $P(n)$ is true: $a_n = 3^n - 2^n$

$$a_n = 5a_{n-1} - 6a_{n-2} \qquad \text{by def. of seq.}$$

$$= 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2}) \qquad \text{by S.I.H.}$$

$$= (3+2)(3^{n-1} - 2^{n-1}) - 3 \cdot 2(3^{n-2} - 2^{n-2})$$

$$= 3 \cdot 3^{n-1} - \cancel{3 \cdot 2^{n-1}} + \cancel{2 \cdot 3^{n-1}} - 2 \cdot 2^{n-1} - \cancel{2 \cdot 3 \cdot 3^{n-2}} + \cancel{3 \cdot 2 \cdot 2^{n-2}}$$

$$= 3 \cdot 3^{n-1} - 2 \cdot 2^{n-1} = 3^n - 2^n$$

Thus, $P(n)$ is shown true for all integers $n \geq 0$ by strong induction.