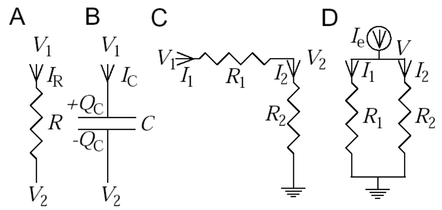
Model Neurons: Neuroelectronics (Part I)

Dayan and Abbott (2001) Chapter 5 and Appendix A.4.

- Basic electrical circuits.
- Passive membrane model.
- Single compartment model.
- Integrate-and-fire neurons.
- Hodgkin-Huxley model.
- Synaptic coductances.

Instructor: Yoonsuck Choe; CPSC 644 Cortical Networks

Electrical Circuits



- Ohm's law:

$$V_R = I_R R,$$

V: voltage, I: current, R: resistence.

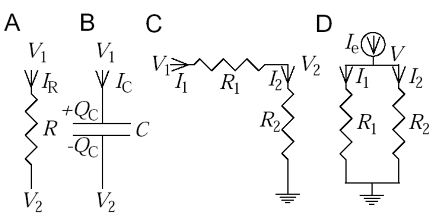
- Charge across a capacitor:

$$CV_C = Q_C$$

$$C\frac{dV_C}{dt} = \frac{dQ_C}{dt} = I_C,$$

V: voltage, Q: charge, I: current.

Electrical Circuits: Serial Resistors



Kirchhoff's current law: At a node, all currents sum to zero (or, sum of incoming = sum of outgoing currents).

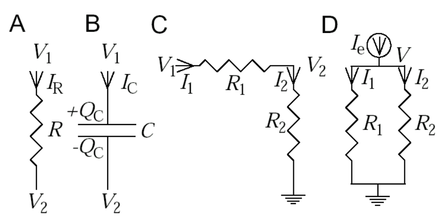
- Example C: at node next to V_2, $I_1 = I_2$. Thus:

$$V_1 - V_2 = I_1 R_1, V_2 - 0 = I_2 R_2$$

$$V_1 = I_1(R_1 + R_2), V_2 = I_2 R_2 = I_1 R_2$$

$$V_2 = \frac{V_1 R_2}{R_1 + R_2}.$$

Electrical Circuits: Parallel Resistors



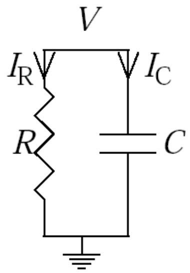
At the node next to V, $I_e = I_1 + I_2$.

$$I_1 = \frac{V}{R_1}, I_2 = \frac{V}{R_2}$$

$$I_e = \frac{V}{R_1} + \frac{V}{R_2} = \frac{R_1 + R_2}{R_1 R_2} V$$

Thus, total resistence of parallel resistors is $\frac{R_1 R_2}{R_1 + R_2}$.

Resistor-Capacitor Circuit (I)



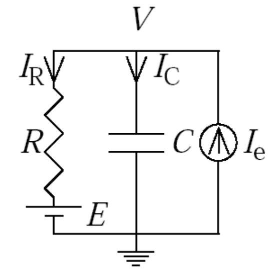
Case A: No external current source: $I_{\mathrm{R}} + I_{\mathrm{C}} = 0$.

$$I_{\mathrm{R}} + I_{\mathrm{C}} = \frac{V - 0}{R} + C\frac{dV}{dt} = 0$$

$$C\frac{dV}{dt} = -\frac{V}{R}$$

which is a homogeneous linear differential equation, and the general solution is (straight-forward integration after separating the variables): $V(t) = V(0)\exp(-t/RC)$.

Resistor-Capacitor Circuit (II)



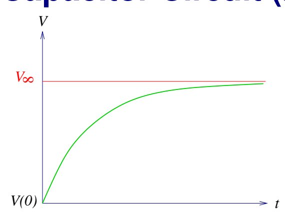
Case B: With external current source: $I_{\mathrm{R}} + I_{\mathrm{C}} = I_{\mathrm{e}}$.

$$I_{\mathrm{R}} + I_{\mathrm{C}} = \frac{V - E}{R} + C\frac{dV}{dt} = I_{\mathrm{e}}$$

$$C\frac{dV}{dt} = \frac{E - V}{R} + I_{\mathrm{e}}$$

which is a nonhomogeneous linear differential equation, and the general solution is: $V(t) = V_\infty + (V(0) - V_\infty)\exp(-t/\tau)$.

Resistor-Capacitor Circuit (II, Cont'd)



The steady state of the membrane equation is:

$$C\frac{dV}{dt} = \frac{E - V}{R} + I_{\mathrm{e}} = 0,$$

$$V = E + I_{\mathrm{e}}R,$$

which we define as $V_\infty = E + I_{\mathrm{e}}R$, and the time constant is $\tau = RC$, which gives the equation in the previous page:

$$V(t) = V_\infty + (V(0) - V_\infty)\exp(-t/\tau).$$

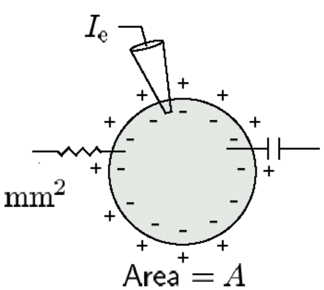
For the solution, first get the general solution V_{h} for the homogeneous case and set $V = V_{\mathrm{h}} \cdot u$, where u is a dummy variable. Solve for V.

Single Compartment Model

$\Delta V = I_{\mathrm{e}}R_{\mathrm{m}}$

$R_{\mathrm{m}} = r_{\mathrm{m}}/A$

$r_{\mathrm{m}} \approx 1\mathrm{M}\Omega \cdot \mathrm{mm}^2$



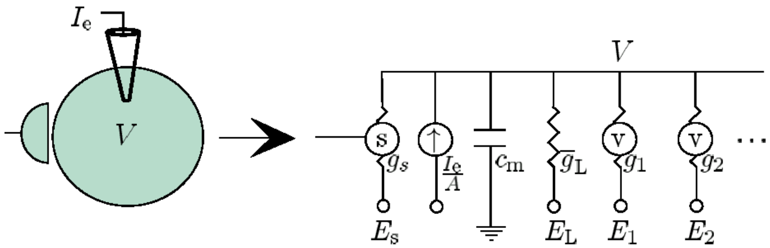
$Q = C_{\mathrm{m}}V$

$C_{\mathrm{m}} = c_{\mathrm{m}}A$

$c_{\mathrm{m}} \approx 10\mathrm{nF}/\mathrm{mm}^2$

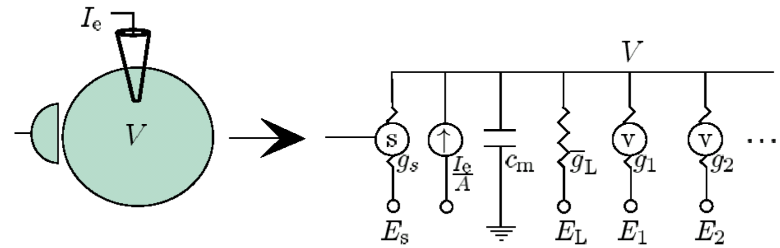
- V: membrane potential
- r_{m}: specific membrane resistance
- c_{m}: specific membrane capacitance
- I_{e}: input current
- Conductance: reciprocal of resistance, denoted g.

Single Compartment Model: Circuit



- Leakage current: $i_L = \bar{g}_L(V - E_L)$.
- Membrane current: $i_m = \sum_i g_i(V - E_i)$.
- Input current: I_e/A.
- Current across capacitor: $c_m \frac{dV}{dt} = I_C$.

Single Compartment Model: Equation



Incoming: I_e/A; Outgoing: all the rest. So, we get:

$$\frac{I_e}{A} = c_m \frac{dV}{dt} + \sum_i g_i(V - E_i),$$

which becomes:

$$c_m \frac{dV}{dt} = -i_m + \frac{I_e}{A}.$$

Integrate and Fire Models

- Basically an RC circuit with the R-part serving as the leakage:

$$c_m \frac{dV}{dt} = -\bar{g}_L(V - E_L) + \frac{I_e}{A}.$$

- Multiplying both sides with r_m gives ($r_m = 1/\bar{g}_L, \tau_m = r_m c_m, R_m = r_m/A$):

$$\tau_m \frac{dV}{dt} = E_L - V + R_m I_e.$$

  When $I_e = 0$, steady state voltage becomes $V = E_L$, which is the resting membrane potential (V_{rest}).

- When V reaches a threshold V_{th}, generate a spike and reset the membrane potential to V_{rest}.

Integrate and Fire Models: Analytic Solution

- Exact solution gives:

$$V(t) = E_L + R_m I_e + (V(0) - E_L - R_m I_e)\exp(-t/\tau_m),$$

  which is the same as in page 7.

- $V_\infty = E_L + R_m I_e$, and this value should be greater than the threshold V_{th} for the neuron to fire at all. Given a fixed E_L and R_m, the only thing that can change V_∞ is then the input current I_e.
- Given a constant input current I_e that allows spiking, the spiking frequency can be analytically calculated.
- First, calculate the time to first spike, when $V(t) = V_{th}$ with $V(0) = V_{rest}$, and solve for t.

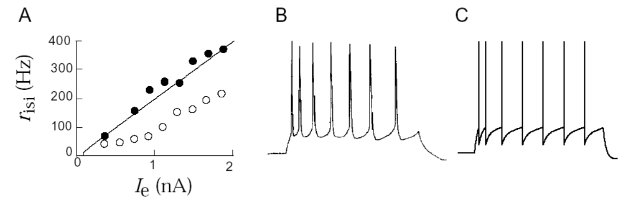
Integrate and Fire Models: Firing Rate

- The calculation comes out to:

$$t_{\mathrm{isi}} = \tau_{\mathrm{m}} \ln \left(\frac{R_{\mathrm{m}} I_{\mathrm{e}} + E_{\mathrm{L}} - V_{\mathrm{rest}}}{R_{\mathrm{m}} I_{\mathrm{e}} + E_{\mathrm{L}} - V_{\mathrm{th}}} \right).$$

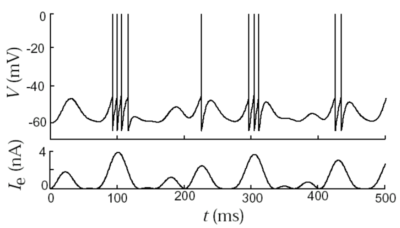
- Since the neuron will fire every t_{isi} time units, this gives the "inter-spike interval" (or ISI).
- Thus, firing occurs with a period of t_{isi}, and so the firing frequency is $r_{\mathrm{isi}} = 1/t_{\mathrm{isi}}$.
- Note again that $V_{\mathrm{th}} < V_{\infty} = E_{\mathrm{L}} + R_{\mathrm{m}} I_{\mathrm{e}}$ must hold. Otherwise, no spikes.

Integrate and Fire Model: Firing Rate



- Plot shows r_{isi} dependent on the input current (in INF vs. real data), and real neuron vs. INF firing.
- Without spike adaptation, INF fits the real data well (black dots).
- Spike adaptation means dynamic change in firing rate as a neuron keeps firing.

Integrate and Fire Model



- INF model with a fluctuating driving input is shown.
- The spikes (the long peaks) are shown just as a visualization, and they are not represented in the equation.
- Usually simple numerical integration is used for the simulation (use Taylor series expansion and drop higher-order terms):

$$\tau_{\mathrm{m}} \frac{\Delta V}{\Delta t} = E_{\mathrm{L}} - V(t) + R_{\mathrm{m}} I_{\mathrm{e}}(t)$$

$$\Delta V = \frac{(E_{\mathrm{L}} - V(t) + R_{\mathrm{m}} I_{\mathrm{e}}(t))}{\tau_{\mathrm{m}}} \Delta t$$

$$V(t + \Delta t) = V(t) + \Delta V.$$