Elementary Differential Equations

- First-order linear differential equations and solutions.
- Numerical solutions.
- Reference: E. Kreyszig, *Advanced Engineering Mathematics*, 5th ed., Wiley, 1983.

First-order Linear Differential Equations

- General form takes:

$$y' + a(x)y = r(x),$$

  where $y' = dy/dx$ and y is a function of x. It is first order since the highest derivative is first order (y'). It is linear, because it is a linear function of y' and y (no terms like y^2 or $y'y$, etc.).

- Homogeneous, when $r(x) = 0$. In this case, the solution can be found easily through *separation of variables*.
- Nonhomogeneous, when $r(x) \neq 0$. The solution in this case is a bit more involved, and several different approaches exist.

Homogeneous 1st-order Linear Diff. Eq.

1. Start with

$$y' + a(x)y = 0$$

2. Subtract $a(x)y$ from both sides:

$$y' = -a(x)y$$

3. Divide both sides by y:

$$\frac{1}{y}\frac{dy}{dx} = -a(x)$$

4. Multiply both sides with dx:

$$\frac{1}{y}dy = -a(x)dx$$

5. Integrate:

$$\int \frac{1}{y}dy = \int -a(x)dx$$

$$\ln y = -\int a(x)dx + c$$

6. Apply $\exp(\cdot)$:

$$y = \exp\left(-\int a(x)dx + c\right)$$

$$y = \exp\left(-\int a(x)dx\right) \times \exp(c)$$

$$y = C \exp\left(-\int a(x)dx\right)$$

This called *separation of variables*.

Nonhomogeneous 1st-order Diff. Eq.

1. Start with

$$y' + a(x)y = r(x)$$

2. Let y_h be the solution to the homogeneous case (where $r(x) = 0$), so that

$$y_h' + a(x)y_h = 0$$

3. For now, assume

$$y(x) = y_h(x)u(x)$$

4. Plug in 3 to 1:

$$\frac{d(y_h(x)u(x))}{dx} + a(x)y_h(x)u(x) = r(x)$$

$$y_h'(x)u(x) + y_h(x)u'(x) + a(x)y_h(x)u(x) = r(x)$$

$$u(x)\underbrace{\left(y_h'(x) + a(x)y_h(x)\right)}_{\text{This is 0, from step 2.}} + y_h(x)u'(x) = r(x)$$

$$y_h(x)u'(x) = r(x)$$

Nonhomogeneous 1st-order Diff. Eq.

5. Continuing from:

$$y_h(x)u'(x) = r(x)$$

6. divide both sides by $y_h(x)$:

$$\frac{du}{dx} = \frac{r(x)}{y_h(x)}$$

7. Multiply both sides with dx:

$$du = \frac{r(x)}{y_h(x)}dx$$

8. Integrate:

$$\int du = \int \frac{r(x)}{y_h(x)}dx$$

$$u = \int \frac{r(x)}{y_h(x)}dx + c$$

9. Plug this in to $y(x) = y_h(x)u(x)$ to get:

$$y(x) = y_h(x)\left(\int \frac{r(x)}{y_h(x)}dx + c\right)$$

This called *variation of parameters*.

Numerical Solution of Diff. Eq. (not so sophisticated)

1. Given $f'(x)$ and the initial condition $f(0)$
2. Take the Taylor series expansion of $f(x)$ around a:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots$$

3. Drop higher order terms:

$$f(x) \approx f(a) + f'(a)(x-a)$$

4. Set $x = t + \Delta t$ and $a = t$ for a small Δt:

$$f(t+\Delta t) \approx f(t) + f'(t)(t + \Delta t - t) = f(t) + f'(t)\Delta t$$

5. Starting with $f(0)$, the approximate values $\hat{f}(\cdot)$ becomes:

$$\hat{f}(\Delta t) = f(0) + f'(0)\Delta t$$

$$\hat{f}(2\Delta t) = \hat{f}(\Delta t) + f'(\Delta t)\Delta t$$

$$\hat{f}(3\Delta t) = \hat{f}(2\Delta t) + f'(2\Delta t)\Delta t \quad \text{and so on...}$$