

# Search and Game Playing

## Overview

- Search problems: definition
- Example: 8-puzzle
- General search
- Evaluation of search strategies
- Strategies: breadth-first, uniform-cost, depth-first
- More uninformed search: depth-limited, iterative deepening, bidirectional search

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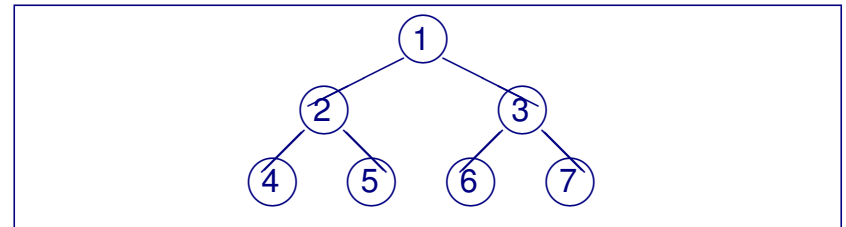
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## Emacs Tips

- multiple windows in emacs (up/down): C-x 2
- multiple windows in emacs (left/right): C-x 3
- switch between buffers: C-x b
- reduce to one window: C-x 1
- navigation between windows in emacs: C-x o
- increasing height of window in emacs: C-x ^
- killing current window in emacs: C-x k

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## Search Problems: Definition



**Search** = < initial state, operators, goal states >

- Initial State: description of the current situation as given in a problem
- Operators: functions from any state to a set of successor (or neighbor) states
- Goal: subset of states, or test rule

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## Variants of Search Problems

**Search** =  $\langle$  state space, initial state, operators, goal states  $\rangle$

- State space: set of all possible states reachable from the current initial state through repeated application of the operators (i.e. path).

**Search** =  $\langle$  initial state, operators, goal states, path cost  $\rangle$

- Path cost: find **the best** solution, not just **a** solution. Cost can be many different things.

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## Search State

State as Data Structure

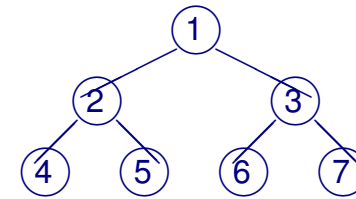
- examples: variable assignment, properties, order in list, bitmap, graph (vertex and edges)
- captures all possible ways world could be
- typically static, discrete (symbolic), but does not have to be

Choosing a Good Representation

- concise (keep only the relevant features)
- explicit (easy to compute when needed)
- embeds constraints

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## Types of Search



- Uninformed: systematic strategies (Chapter 3)
- Informed: Use domain knowledge to narrow search (Chapter 4)
- Game playing as search: minimax, state pruning, probabilistic games (Chapter 5).

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## Operators

Function from state to subset of states

- drive to neighboring city
- place piece on chess board
- add person to meeting schedule
- slide tile in 8-puzzle

Characteristics

- often requires instantiation (fill in variables)
- encode constraints (only certain operations are allowed)
- generally discrete: continuous parameters  $\rightarrow$  infinite branching

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## Goals: Subset of states or test rules

Specification:

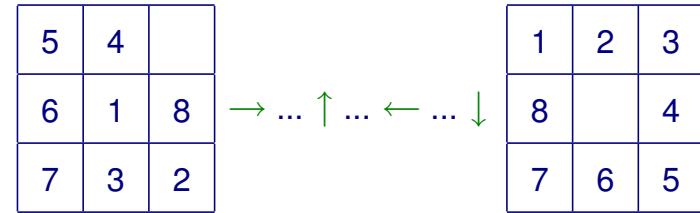
- set of states: enumerate the eligible states
- partial description: e.g. a certain variable has value over  $x$ .
- constraints: or set of constraints. Hard to enumerate all states matching the constraints, or very hard to come up with a solution at all (i.e. you can only verify it; P vs. NP).

Other considerations:

- space, time, quality (exact vs. approximate trade-offs)

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## An Example: 8-Puzzle

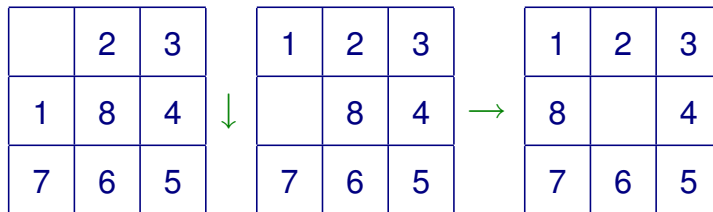


- **State:** location of 8 number tiles and one blank tile
- **Operators:** blank moves left, right, up, or down
- **Goal test:** state matches the configuration on the right (see above)
- **Path cost:** each step cost 1, i.e. path length, or search tree depth

Generalization: 15-puzzle, ...,  $(N^2 - 1)$ -puzzle

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## 8-Puzzle: Example



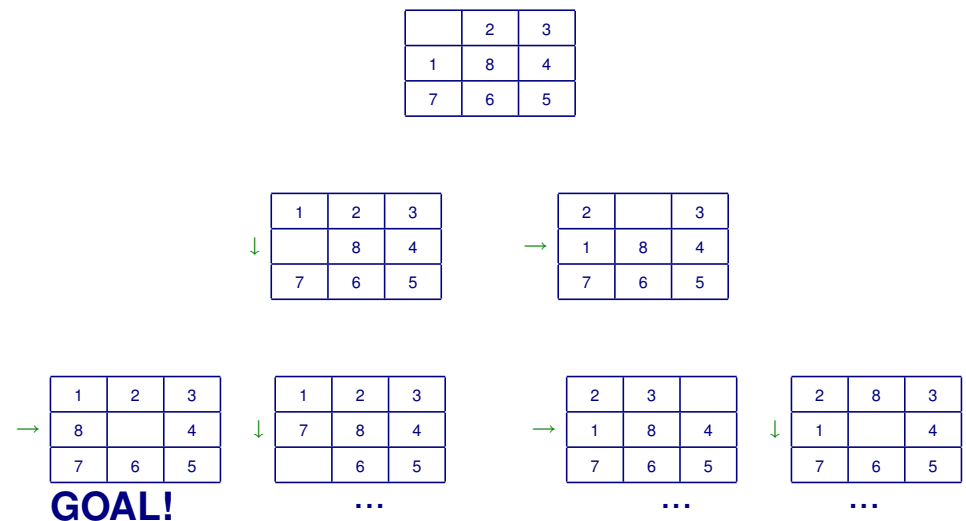
Possible state representations in LISP (0 is the blank):

- (0 2 3 1 8 4 7 6 5)
- ((0 2 3) (1 8 4) (7 6 5))
- ((0 1 7) (2 8 6) (3 4 5))
- or use the `make-array`, `aref` functions.

How easy to: (1) compare, (2) operate on, and (3) store (i.e. size).

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## 8-Puzzle: Search Tree



**GOAL!**

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## Goal Test

As simple as a single LISP call:

```
* (defvar *goal-state* '(1 2 3 8 0 4 7 6 5))
*GOAL-STATE*

* (equal *goal-state* '(1 2 3 8 0 4 7 6 5))
T
```

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## Evaluation of Search Strategies

- time-complexity: how many nodes expanded so far?
- space-complexity: how many nodes must be stored in node-list at any given time?
- completeness: if solution exists, guaranteed to be found?
- optimality: guaranteed to find the best solution?

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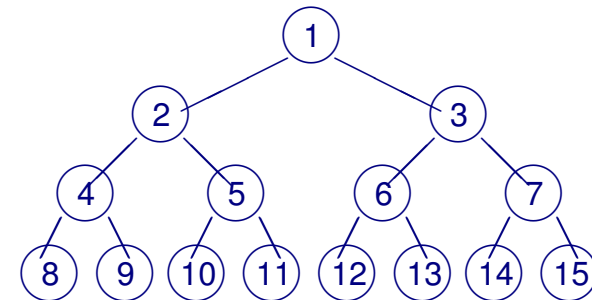
## General Search Algorithm

Pseudo-code:

```
function General-Search (problem, Que-Fn)
  node-list := initial-state
  loop begin
    // fail if node-list is empty
    if Empty(node-list) then return FAIL
    // pick a node from node-list
    node := Get-First-Node(node-list)
    // if picked node is a goal node, success!
    if (node == goal) then return as SOLUTION
    // otherwise, expand node and enqueue
    node-list := Que-Fn(node-list, Expand(node))
  loop end
```

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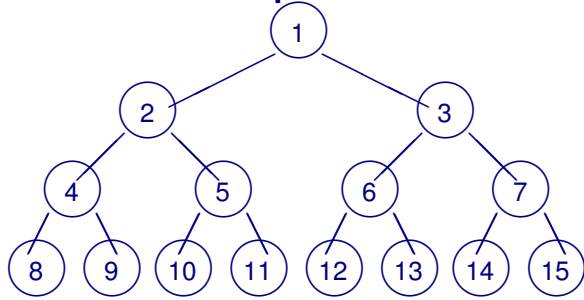
## Breadth First Search



- node visit order (goal test): 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
- queuing function: enqueue at end (add expanded node at the end of the list)

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## BFS: Expand Order



Evolution of the queue (**bold**= expanded and added children):

1. **[1]** : initial state
2. **[2][3]** : dequeue 1 and enqueue 2 and 3
3. [3] **[4][5]** : dequeue 2 and enqueue 4 and 5
4. [4] [5] **[6][7]** : **all** depth 3 nodes
- ...
8. [8] [9] [10] [11] [12] [13] **[14][15]** : **all** depth 4 nodes

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## Uniform Cost

BFS with expansion of lowest-cost nodes: path cost is  $g(\text{node})$ .

- BFS:  $g(n) = \text{Depth}(\text{node})$

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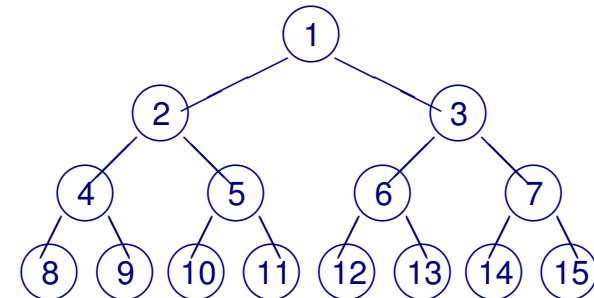
## BFS: Evaluation

branching factor  $b$ , depth of solution  $d$ :

- complete: it will find the solution if it exists
- time:  $1 + b + b^2 + \dots + b^d$
- space:  $O(b^{d+1})$  where  $d$  is the depth of the shallowest solution
- space is more problem than time in most cases (p 75, figure 3.12).
- time is also a major problem nonetheless (same as time)

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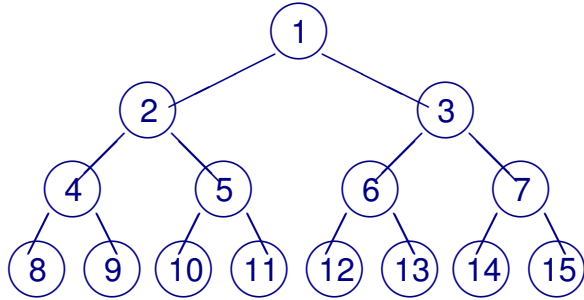
## Depth First Search



- node visit order (goal test): 1 2 4 8 9 5 10 11 3 6 12 13 7 14 15
- queuing function: enqueue at left (stack push; add expanded node at the beginning of the list)

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## DFS: Expand Order



Evolution of the queue (**bold**=expanded and added children):

1. [ 1 ] : initial state
2. **[2][3]** : pop 1 and push expanded in the front
3. **[4][5]** [ 3 ] : pop 2 and push expanded in the front
4. **[8][9]** [ 5 ] [ 3 ] : pop 4 and push expanded in the front

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## Implementation

- Use of stack or queue : explicit storage of expanded nodes
- Recursion : implicit storage in the recursive call stack

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## DFS: Evaluation

branching factor  $b$ , depth of solutions  $d$ , max depth  $m$ :

- incomplete: may wander down the wrong path
- time:  $O(b^m)$  nodes expanded (worst case)
- space:  $O(bm)$  (just along the current path)
- good when there are many shallow goals
- bad for deep or infinite depth state space

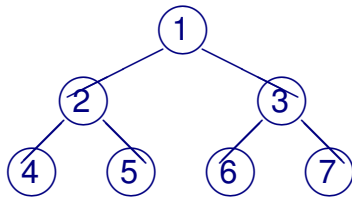
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## Key Points

- Description of a search problem: initial state, goals, operators, etc.
- Considerations in designing a representation for a state
- Evaluation criteria
- BFS, UCS, DFS: time and space complexity, completeness
- Differences and similarities between BFS and UCS
- When to use one vs. another
- Node visit orders for each strategy
- Tracking the stack or queue at any moment

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## Depth Limited Search (DLS): Limited Depth DFS



- node visit order for each depth limit  $l$ :  
1 ( $l = 1$ ); 1 2 3 ( $l = 2$ ); 1 2 4 5 3 6 7 ( $l = 3$ );
- queuing function: enqueue at front (i.e. stack push)
- push the depth of the node as well:**  
( $\langle \text{depth} \rangle \langle \text{node} \rangle$ )

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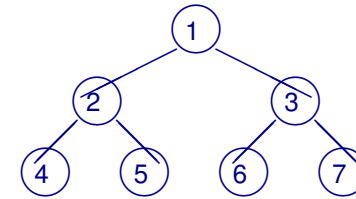
## DLS: Evaluation

branching factor  $b$ , depth limit  $l$ , depth of solution  $d$ :

- complete: if  $l \geq d$
- time:  $O(b^l)$  nodes expanded (worst case)
- space:  $O(bl)$  (same as DFS, where  $l = m$  ( $m$ : max depth of tree in DFS))
- good if solution is within the limited depth.
- non-optimal (same problem as in DFS).

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## DLS: Expand Order



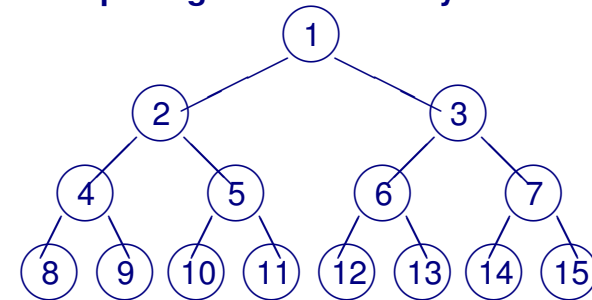
Evolution of the queue (**bold**=expanded and then added):

- ( $\langle \text{depth} \rangle, \langle \text{node} \rangle$ ); Depth limit = 3
- [ ( $d_1, 1$ ) ] : initial state
  - [ **( $d_2, 2$ )** ] [ **( $d_2, 3$ )** ] : pop 1 and push 2 and 3
  - [ **( $d_3, 4$ )** ] [ **( $d_3, 5$ )** ] [ ( $d_2, 3$ ) ] : pop 2 and push 4 and 5
  - [ ( $d_3, 5$ ) ] [ ( $d_2, 3$ ) ] : pop 4, cannot expand it further
  - [ ( $d_2, 3$ ) ] : pop 5, cannot expand it further
  - [ **( $d_3, 6$ )** ] [ **( $d_3, 7$ )** ] : pop 3, and push 6, 7

...

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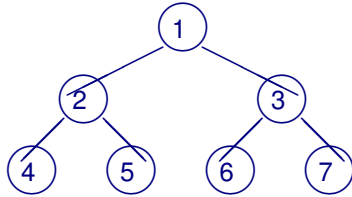
## Iterative Deepening Search: DLS by Increasing Limit



- node visit order:  
1 ; 1 2 3; 1 2 4 5 3 6 7; 1 2 4 8 9 5 10 11 3 6 12 13 7 14 15; ...
- revisits already explored nodes at successive depth limit
- queuing function: enqueue at front (i.e. stack push)
- push the depth of the node as well:** ( $\langle \text{depth} \rangle \langle \text{node} \rangle$ )

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## IDS: Expand Order



Basically the same as DLS: Evolution of the queue (**bold**=expanded and then added): ( $\langle \text{depth}, \langle \text{node} \rangle \rangle$ ); e.g. Depth limit = 3

1. [ (d1, 1) ] : initial state
2. [(d2,2)][(d2,3)] : pop 1 and push 2 and 3
3. [(d3,4)][(d3,5)] [ (d2, 3) ] : pop 2 and push 4 and 5
4. [ (d3, 5) ] [ (d2, 3) ] : pop 4, cannot expand it further
5. [ (d2, 3) ] : pop 5, cannot expand it further
6. [(d3,6)][(d3,7)]: pop 3, and push 6, 7

...

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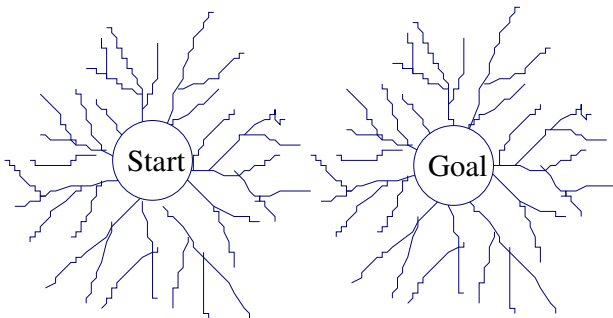
## IDS: Evaluation

branching factor  $b$ , depth of solution  $d$ :

- complete: cf. DLS, which is conditionally complete
- time:  $O(b^d)$  nodes expanded (worst case)
- space:  $O(bd)$  (cf. DFS and DLS)
- **optimal!**: unlike DFS or DLS
- good when search space is huge and the depth of the solution is not known (\*)

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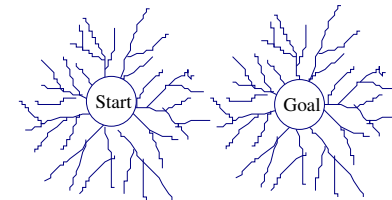
## Bidirectional Search (BDS)



- Search from both initial state and goal to reduce search depth.
- $O(b^{d/2})$  of BDS vs.  $O(b^{d+1})$  of BFS.

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## BDS: Considerations

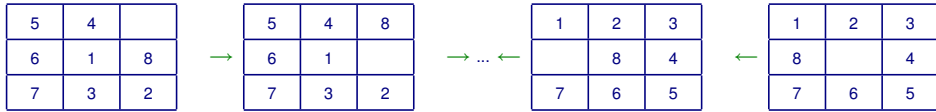


1. how to back trace from the goal?
2. successors and predecessors: are operations reversible?
3. are goals explicit?: need to know the goal to begin with
4. check overlap in two branches
5. BFS? DFS? which strategy to use? Same or different?

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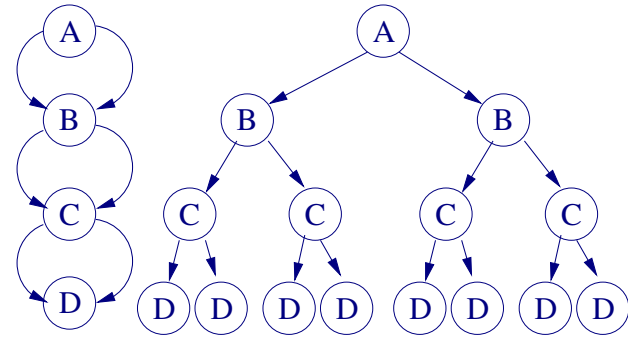
## BDS Example: 8-Puzzle



- Is it a good strategy?
- What about Chess? Would it be a good strategy?
- What kind of domains may be suitable for BDS?

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## Avoiding Repeated States

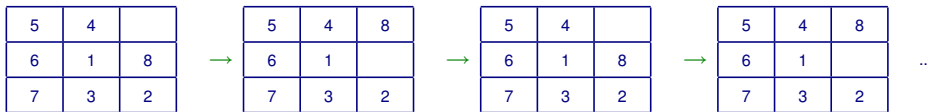


Repeated states can be devastating in search problems.

- Common cases: problems with reversible operators → search space becomes infinite
- One approach: find a spanning tree of the graph

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## Avoiding Repeated States: Strategies



- Do not return to the node's parent
- Avoid cycles in the path (this is a huge theoretical problem in its own right)
- Do not generate states that you generated before: use a hash table to make checks efficient

How to avoid storing every state? Would using a short signature (or a checksum) of the full state description help?

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## Key Points

- DLS, IDS, BDS search order, expansions, and queuing
- DLS, IDS, BDS evaluation
- DLS, IDS, BDS: suitable domains
- Repeated states: why removing them is important

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## Overview

- Best-first search
- Heuristic function
- Greedy best-first search
- $A^*$
- Designing good heuristics
- $IDA^*$
- Iterative improvement algorithms
  1. Hill-climbing
  2. Simulated annealing

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## Best First Search

```
function Best-First-Search (problem, Eval-Fn)
    Queuing-Fn ← sorted list by Eval-Fn(node)
    return General-Search(problem, Queuing-Fn)
```

- The queuing function queues the expanded nodes, and sorts it every time by the  $Eval-Fn$  value of each node.
- One of the simplest  $Eval-Fn$ : **estimated cost** to reach the goal.

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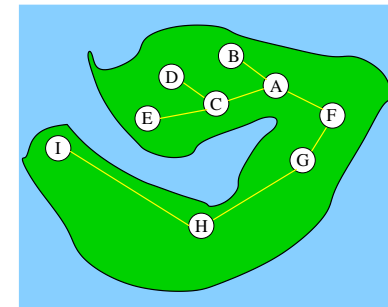
## Informed Search (Chapter 4)

From domain knowledge, obtain an **evaluation function**.

- best-first search: order nodes according to the evaluation function value
- greedy search: minimize estimated cost for reaching the goal – fast, but incomplete and non-optimal.
- $A^*$ : minimize  $f(n) = g(n) + h(n)$ , where  $g(n)$  is the current path cost from start to  $n$ , and  $h(n)$  is the estimated cost from  $n$  to goal.

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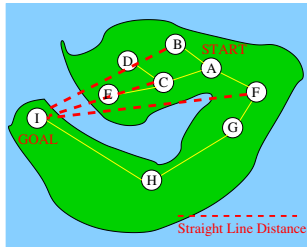
## Heuristic Function



- $h(n)$  = estimated cost of the cheapest path from the state at node  $n$  to a goal state.
- The only requirement is the  $h(n) = 0$  at the goal.
- **Heuristics** means “to find” or “to discover”, or more technically, “how to solve problems” (Polya, 1957).

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## Heuristics: Example



- $h_{SLD}(n)$ : straight line distance (SLD) is one example.
- Start from **A** and Goal is **I**: **C** is the most promising next step in terms of  $h_{SLD}(n)$ , i.e.  $h(C) < h(B) < h(F)$
- Requires some knowledge:
  1. coordinates of each city
  2. generally, cities toward the goal tend to have smaller **SLD**.

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## Greedy Best-First Search

**function** Greedy-Best-First Search (*problem*)

$h(n)$ =estimated cost from  $n$  to goal

**return** Best-First-Search(*problem*, $h$ )

- Best-first with heuristic function  $h(n)$

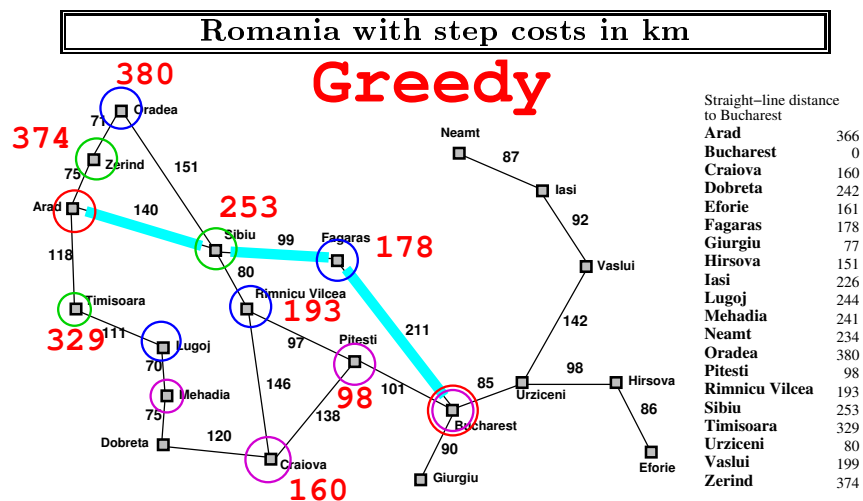
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## Greedy Best-First Search: Evaluation

Branching factor  $b$  and max depth  $m$ :

- Fast, just like Depth-First-Search: single path toward the goal.
- Time:  $O(b^m)$
- Space: same as time – all nodes are stored in sorted list(!), **unlike DFS**
- Incomplete, just like DFS
- Non-optimal, just like DFS

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Total Path Cost = 450

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## A\* : Uniform Cost + Heuristic Search

Avoid expanding paths that are already found to be expensive:

- $f(n) = g(n) + h(n)$
- $f(n)$  : estimated cost to goal through node  $n$
- **provably complete and optimal!**
- **restrictions:**  $h(n)$  should be an **admissible heuristic**
- admissible heuristic: one that **never overestimate** the actual cost of the best solution through  $n$

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## Behavior of A\* Search

- usually, the  $f$  value never decreases along a given path:  
**monotonicity**
- in case it is nonmonotonic, i.e.  $f(Child) < f(Parent)$ ,  
make this adjustment:  
 $f(Child) = \max(f(Parent), g(Child) + h(Child))$ .
- this is called **pathmax**

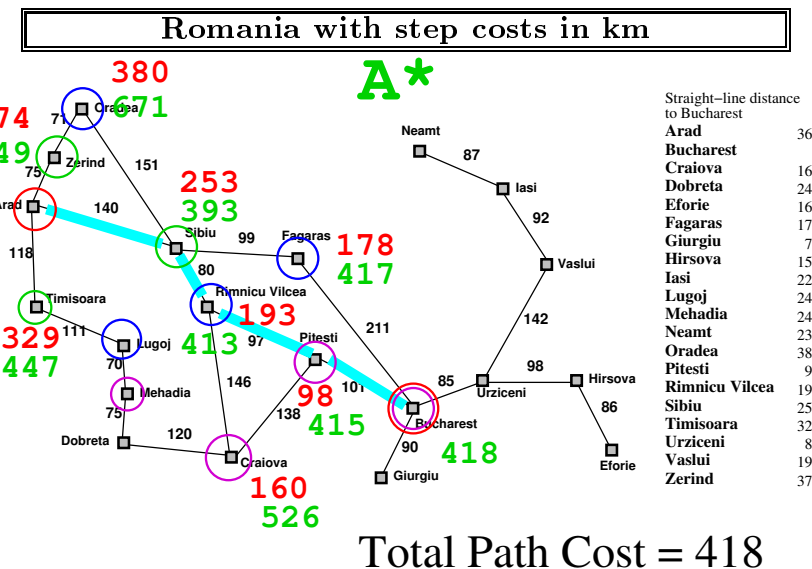
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## A\* Search

```
function A*-Search (problem)
    g(n)=current cost up till n
    h(n)=estimated cost from n to goal
    return Best-First-Search(problem,g + h)
```

- Condition:  $h(n)$  must be an **admissible heuristic function!**
- A\* is **optimal!**

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## Optimality of A\*

$G_2$ : suboptimal goal in the node-list.

$n$ : unexpanded node on a shortest path to goal  $G_1$

- $f(G_2) = g(G_2)$  since  $h(G_2) = 0$
- $> g(G_1)$  since  $G_2$  is suboptimal
- $\geq f(n)$  since  $h$  is admissible

Since  $f(G_2) > f(n)$ , A\* will never select  $G_2$  for expansion.

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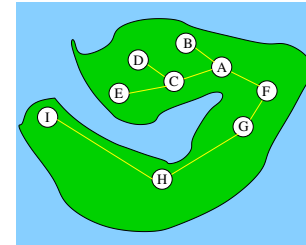
## Lemma to Optimality of A\*

Lemma: A\* expands nodes in order of increasing  $f(n)$  value.

- Gradually adds **f-contours** of nodes (cf. BFS adds layers).
- The goal state may have a  $f$  value: let's call it  $f^*$
- This means that all nodes with  $f < f^*$  will be expanded!

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## Optimality of A\*: Example



1. **Expansion of parent allowed:** search fails at nodes **B**, **D**, and **E**.
2. **Expansion of parent disallowed:** paths through nodes **B**, **D**, and **E** will have an inflated path cost  $g(n)$ , thus will become nonoptimal.

$A \rightarrow C \rightarrow E \rightarrow C \rightarrow A \rightarrow F \rightarrow \dots$   
 inflated path cost

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## Complexity of A\*

A\* is complete and optimal, but space complexity can become exponential if the heuristic is not good enough.

- condition for **subexponential** growth:

$$|h(n) - h^*(n)| \leq O(\log h^*(n)),$$

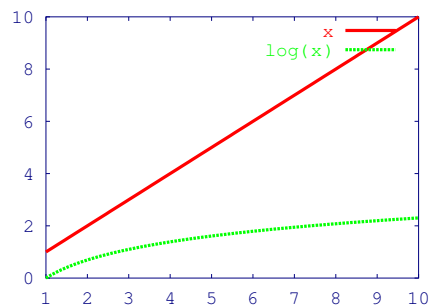
where  $h^*(n)$  is the **true** cost from  $n$  to the goal.

- that is, error in the estimated cost to reach the goal should be less than even linear, i.e.  $< O(h^*(n))$ .

Unfortunately, with most heuristics, error is at least proportional with the true cost, i.e.  $\geq O(h^*(n)) > O(\log h^*(n))$ .

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## Linear vs. Logarithmic Growth Error



- Error in heuristic:  $|h(n) - h^*(n)|$ .
- For most heuristics, the error is at least linear.
- For  $A^*$  to have subexponential growth, the error in the heuristic should be on the order of  $O(\log h^*(n))$ .

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## $A^*$ : Evaluation

- Complete : unless there are infinitely many nodes with  $f(n) \leq f(G)$
- Time complexity: exponential in (relative error in  $h \times$  length of solution)
- Space complexity: same as time (keep all nodes in memory)
- Optimal

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## Problem with $A^*$

Space complexity is usually **exponential!**

- we need a memory bounded version
- one solution is: Iterative Deepening  $A^*$ , or  $IDA^*$

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## Heuristic Functions: Example

Eight puzzle

5	4		1	2	3
6	1	8	8		4
7	3	2	7	6	5

- $h_1(n)$  = number of misplaced tiles
  - $h_2(n)$  = total **Manhattan** distance (city block distance)
- $h_1(n) = 7$  (not counting the blank tile)
- $h_2(n) = 2+3+3+2+4+2+0+2 = 18$

\* Both are admissible heuristic functions.

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## Dominance

If  $h_2(n) \geq h_1(n)$  for all  $n$  and both are admissible, then we say that  $h_2(n)$  **dominates**  $h_1(n)$ , and is better for search.

Typical search costs for depth  $d = 14$ :

- Iterative Deepening : 3,473,941 nodes expanded
- $A^*(h_1)$ : 539 nodes
- $A^*(h_2)$ : 113 nodes

Observe that in  $A^*$ , every node with  $f < f^*$  is expanded. Since  $f = g + h$ , nodes with  $h(n) < f^* - g(n)$  will be expanded, so larger  $h$  will result in less nodes being expanded.

- $f^*$  is the  $f$  value for the optimal solution path.

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## Other Heuristic Design

- Use composite heuristics:  $h(n) = \max(h_1(n), \dots, h_m(n))$
- Use statistical information: random sample  $h$  and true cost to reach goal. Find out how often  $h$  and true cost is related.

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## Designing Admissible Heuristics

**Relax the problem** to obtain an admissible heuristics.

For example, in 8-puzzle:

- allow tiles to move anywhere  $\rightarrow h_1(n)$
- allow tiles to move to any adjacent location  $\rightarrow h_2(n)$

For traveling:

- allow traveler to travel by air, not just by road: **SLD**

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## Iterative Deepening $A^*$ : $IDA^*$

$A^*$  is complete and optimal, but the performance is limited by the available space.

- Basic idea: only search within a certain  $f$  bound, and gradually increase the  $f$  bound until a solution is found.
- More on  $IDA^*$  next time.

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## IDA\*

```
function IDA*(problem)
    root ← Make-Node(Initial-State(problem))
    f-limit ← f-Cost(root)
    loop do
        solution, f-limit ← DFS-Contour(root, f-limit)
        if solution != NULL then return solution
        if f-limit == ∞ then return failure
    end loop
```

Basically, iterative deepening depth-first-search with depth defined as the  $f$ -cost ( $f = g + n$ ):

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## IDA\*: Evaluation

- complete and optimal (with same restrictions as in A\*)
- space: proportional to longest path that it explores (because it is depth first!)
- time: dependent on the number of different values  $h(n)$  can assume.

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## DFS-Contour(*root*, *f-limit*)

Find solution from node **root**, within the  $f$ -cost limit of **f-limit**.

DFS-Contour returns **solution sequence** and **new  $f$ -cost limit**.

- if  $f$ -cost(**root**) > **f-limit**, return fail.
- if **root** is a goal node, return solution and new  $f$ -cost limit.
- **recursive call** on all successors and return solution and **minimum  $f$ -limit** returned by the calls
- return **null solution** and new  $f$ -limit by default

Similar to the recursive implementation of DFS.

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## IDA\*: Time Complexity

Depends on the heuristics:

- small number of possible heuristic function values → small number of  $f$ -contours to explore → becomes similar to A\*
- complex problems: each  $f$ -contour only contain one new node  
if A\* expands  $N$  nodes,  
IDA\* expands  
$$1 + 2 + \dots + N = \frac{N(N+1)}{2} = O(N^2)$$
- a possible solution is to have a **fixed** increment  $\epsilon$  for the  $f$ -limit → solution will be suboptimal for at most  $\epsilon$  ( $\epsilon$ -admissible)

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## Other Methods: Beam Search

Best-first search with a fixed limited branching factor

- expand the first  $n$  nodes with the best Eval-Fn value, where  $n$  is a small number.
- $n$  is called the **width of the beam**
- good for domains with continuous time functions (like speech recognition)
- good for domains with huge branching factor (like above)

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## Hill-Climbing

- no queue, keep only the best node
- greedy, no back-tracking
- good for domains where **all nodes are solutions**:
  - goal is to **improve** quality of the solution
  - optimization problems
- note that it is different from greedy search, which keeps a node list

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## Iterative Improvement Algorithms

Start with a complete configuration (all variable values assigned, and **optimal**), and **gradually improve** it.

- Hill-climbing (maximize cost function)
- Gradient descent (minimize cost function)
- Simulated Annealing (probabilistic)

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## Hill-Climbing Strategies

Problems of local maxima, plateau, and ridges:

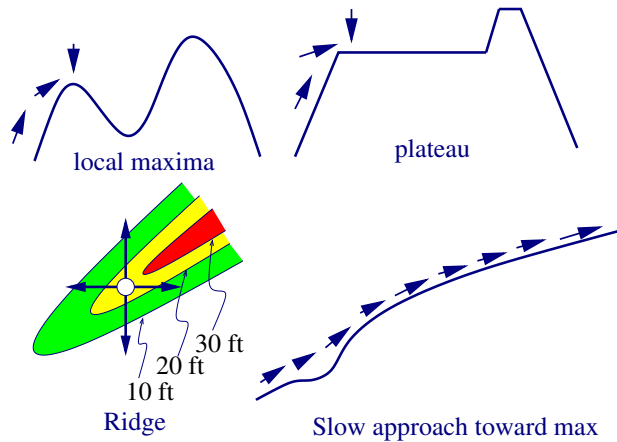
- try **random-restart**: move to a random location in the landscape and restart search from there
- keep  $n$  best nodes (beam search) \*
- parallel search
- simulated annealing \*

Hardness of problem depends on the **shape of the landscape**.

\*: coming up next

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## Hill-Climbing: Problems



- Possible solution: **simulated annealing** – gradually decrease randomness of move to attain globally optimal solution (more on this next week).

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## Simulated Annealing (SA)

Goal: minimize the energy  $E$ , as in statistical thermodynamics.

For successors of the current node,

- if  $\Delta E \leq 0$ , the move is accepted
- if  $\Delta E > 0$ , the move is accepted with probability  $P(\Delta E) = e^{-\frac{\Delta E}{kT}}$ , where  $k$  is the Boltzmann constant and  $T$  is temperature.
- randomness is in the comparison:  $P(\Delta E) < \text{rand}(0, 1)$

$$\Delta E = E_{\text{new}} - E_{\text{old}}$$

The heuristic  $h(n)$  or  $f(n)$  represents  $E$ .

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## Simulated Annealing: Overview

Annealing:

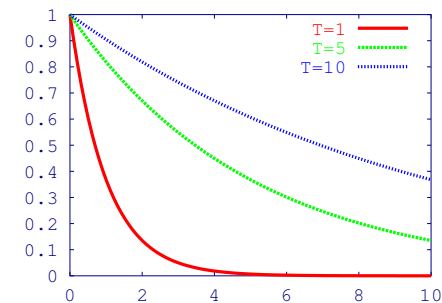
- heating metal to a high-temperature (making it a liquid) and then allowing to cool slowly (into a solid); this relieves internal stresses and results in a more stable, lower-energy state in the solid.
- at high temperature, atoms move actively (large distances with greater randomness), but as temperature is lowered, they become more static.

Simulated annealing is similar:

- basically, hill-climbing with randomness that allows going **down** as well as the standard **up**
- randomness (as temperature) is reduced over time

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## Temperature and $P(\Delta E) < \text{rand}(0, 1)$



Downward moves of any size are allowed at high temperature, but at low temperature, only small downward moves are allowed.

- Higher temperature  $T \rightarrow$  higher probability of **downward** hill-climbing
- Lower  $\Delta E \rightarrow$  higher probability of **downward** hill-climbing

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## T Reduction Schedule

High to low temperature reduction schedule is important:

- reduction too fast: suboptimal solution
- reduction too slow: wasted time
- question: does the form of the reduction schedule curve matter?  
linear, quadratic, exponential, etc.?

The proper values are usually found experimentally.

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## Constraint Satisfaction Search

Constraint Satisfaction Problem (CSP):

- **state:** values of a set of **variables**
- **goal:** test if a set of constraints are met
- **operators:** set values of variables
- general search can be used, but specialized solvers for CSP work better

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## Simulated Annealing Applications

- VLSI wire routing and placement
- Various scheduling optimization tasks
- Traffic control
- Neural network training
- etc.

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## Constraints

- Unary, binary, and higher order constraints: how many variables should simultaneously meet the constraint
- Absolute constraints vs. preference constraints
- Variables are defined in a certain **domain**, which determines the possible set of values, either discrete or continuous.

This is part of a much more complex problem called **constrained optimization problems** in operations research consisting of cost function (either minimize or maximize) and several constraints. Problems can be linear, nonlinear, convex, nonconvex, etc. Straight-forward solutions exist for a limited subclass of these (for example, for linear programming problems can be solved by the simplex method).

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## CSP: continued

- CSPs include NP-complete problems such as 3-SAT, thus finding the solutions can require exponential time.
- However, constraints can help narrow down the possible options, therefore reducing the branching factor. This is because in CSP, the goal can be decomposed into several constraints, rather than being a whole solution.
- Strategies: backtracking (back up when constraint is violated), forward checking (do not expand further if look-ahead returns a constraint violation). Forward checking is often faster and simple to implement.

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## Key Points

- best-first-search: definition
- heuristic function  $h(n)$ : what it is
- greedy search: relation to  $h(n)$  and evaluation. How it is different from DFS (time complexity, space complexity)
- $A^*$ : definition, evaluation, conditions of optimality
- complexity of  $A^*$ : relation to error in heuristics
- designing good heuristics: several rule-of-thumbs
- $IDA^*$ : evaluation, time and space complexity (worst case)
- beam search concept
- hill-climbing concept and strategies
- simulated annealing: core algorithm, effect of  $T$  and  $\Delta E$ , source of randomness.

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## Heuristics for Constraint Satisfaction Problems

General strategies for **variable selection**:

- Most-constrained-variable heuristic (var with fewest possible values)
- Most-constraining-variable heuristic (var involved in the largest number of constraints)

and for **value assignment**:

- Least-constraining-value heuristic (value that rules out the smallest number of values for vars)

Reducing branching factor vs. leaving freedom for future choices.

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## Emacs Tips

**M-x** : [Alt]-[x] or [ESC] then [x], **C-x**: [CTRL]-[x]

- M-x shell (run shell within emacs)
- C-p ( $\uparrow$ ), C-n ( $\downarrow$ ), C-b ( $\leftarrow$ ), C-f ( $\rightarrow$ )
- C-x C-f (load file)
- M-x lisp-mode (environment for editing lisp code)
- C-s (search forward) C-r (reverse search)
- C-g (abort current command in scratch)
- C-k (cut line) C-y (yank, or paste)
- C-space (begin block) C-x C-x (end block) C-w (cut) C-y (yank, or paste)
- C-x u or M-x undo (undo) ; C-x C-s (save) ; C-x C-c (exit)

Full reference card: <http://www.cs.tamu.edu/faculty/choe/courses/02spring/refs>

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# Game Playing

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## Games vs. Search Problems

“Unpredictable” opponent → solution is a contingency plan

Time limits → unlikely to find goal, must approximate

Plan of attack:

- algorithm for perfect play (Von Neumann, 1944)
- finite horizon, approximate evaluation (Zuse, 1945; Shannon, 1950; Samuel, 1952–57)
- pruning to reduce costs (McCarthy, 1956)

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# Game Playing

- attractive AI problem because it is **abstract**
- one of the oldest domains in AI
- in most cases, the world state is fully accessible
- computer representation of the situation can be clear and exact
- challenging: uncertainty introduced by the opponent and the complexity of the problem (full search is impossible)
- hard: in chess, branching factor is about 35, and 50 moves by each player =  $35^{100}$  nodes to search
  - compare to  $10^{40}$  possible legal board states
- *game playing is more like real life than mechanical search*

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## Types of Games

	deterministic	chance
perfect info	chess, checkers, go, othello	backgammon, monopoly
imperfect info	?	bridge, poker, scrabble

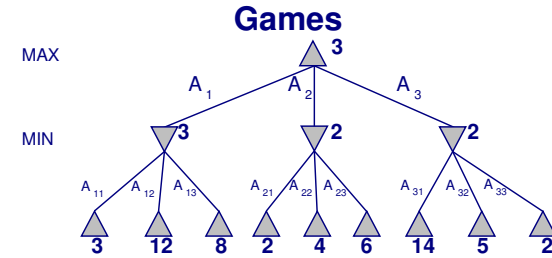
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## Two-Person Perfect Information Game

**initial state:** initial position and who goes first  
**operators:** legal moves  
**terminal test:** game over?  
**utility function:** outcome (win:+1, lose:-1, draw:0, etc.)

- two players (**MIN** and **MAX**) taking turns to maximize their chances of winning (each turn generates one **ply**)
- one player's victory is another's defeat
- need a **strategy** to win no matter what the opponent does

## Minimax: Strategy for Two-Person Perfect Info



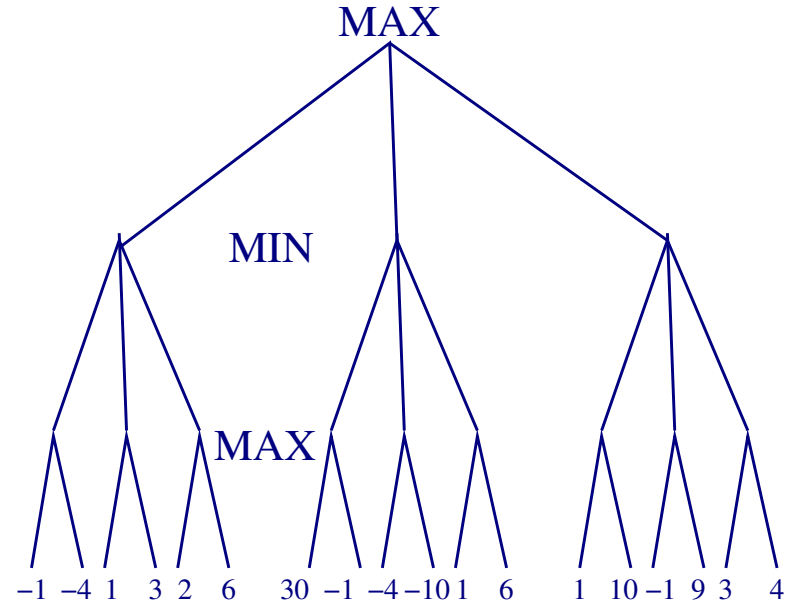
- generate the whole tree, and apply util function to the leaves
- go back upward assigning utility value to each node
- at MIN node, assign **min(successors' utility)**
- at MAX node, assign **max(successors' utility)**
- **assumption:** the opponent acts optimally

## Minimax Decision

**function** Minimax-Decision (game) **returns** operator  
**return** operator that leads to a child state with the **max**(Minimax-Value(child state,game))

**function** Minimax-Value(state,game) **returns** utility value  
**if** Goal(state), **return** Utility(state)  
**else if** Max's move **then**  
 → **return** max of successors' Minimax-Value  
**else**  
 → **return** min of successors' Minimax-Value

## Minimax Exercise



## Minimax: Evaluation

Branching factor  $b$ , max depth  $m$ :

- **complete**: if the game tree is finite
- **optimal**: if opponent is optimal
- **time**:  $b^m$
- **space**:  $bm$  – depth-first (only when utility function values of all nodes are known!)

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## Evaluation Functions

For chess, usually a **linear** weighted sum of feature values:

- $Eval(s) = \sum_i w_i f_i(s)$
- $f_i(s) = (\text{number of white piece X}) - (\text{number of black piece X})$
- other features: degree of control over the center area
- exact values do not matter: the **order** of Minimax-Value of the successors matter.

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## Resource Limits

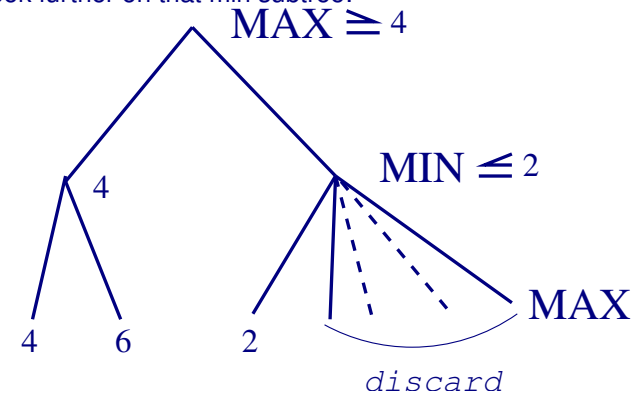
- Time limit: as in Chess  $\rightarrow$  can only evaluate a fixed number of paths
- Approaches:
  - **evaluation function** : how desirable is a given state?
  - **cutoff test** : depth limit
  - **pruning**

Depth limit can result in the **horizon effect**: interesting or devastating events can be just over the horizon!

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## $\alpha$ Cuts

When the current max value is greater than the successor's min value, don't look further on that min subtree:

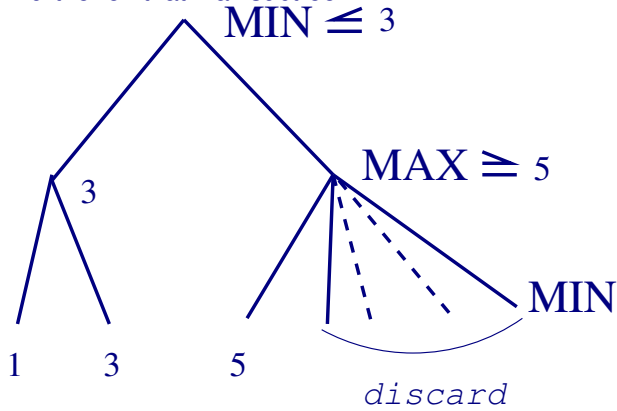


Right subtree can be **at most 2**, so MAX will always choose the left path regardless of what appears next.

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## $\beta$ Cuts

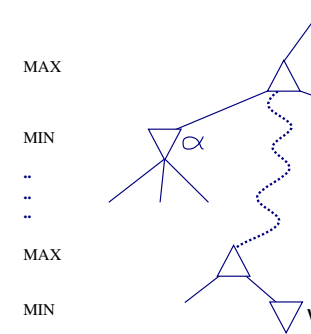
When the current min value is less than the successor's max value, don't look further on that max subtree:



Right subtree can be **at least 5**, so MIN will always choose the left path regardless of what appears next.

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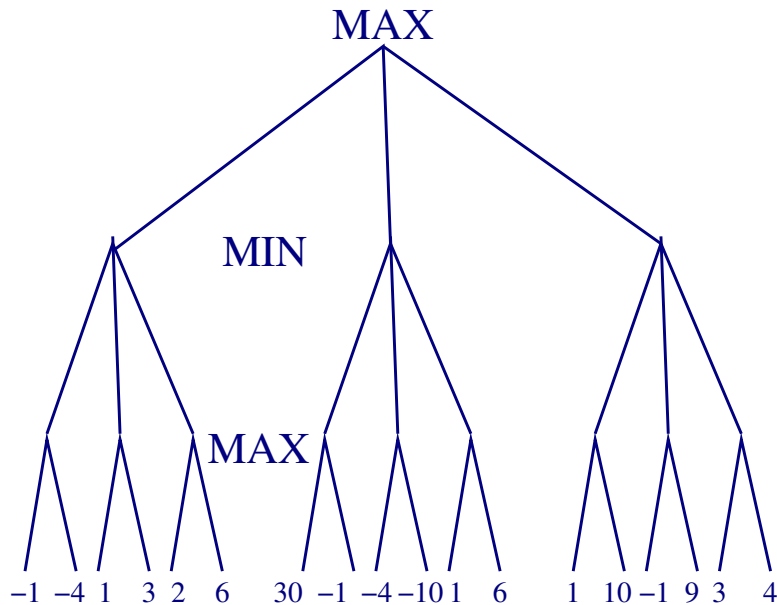
## $\alpha - \beta$ Pruning



- memory of best MAX value  $\alpha$  and best MIN value  $\beta$
- do not go further on any one that does worse than the remembered  $\alpha$  and  $\beta$

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## $\alpha - \beta$ Exercise



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## $\alpha - \beta$ Pruning Properties

Cut off nodes that are known to be suboptimal.

Properties:

- pruning **does not** affect final result
- good move ordering improves effectiveness of pruning
- with **perfect ordering**, time complexity =  $b^{m/2}$ 
  - **doubles** depth of search
  - can easily reach 8-ply in chess
- $b^{m/2} = (\sqrt{b})^m$ , thus  $b = 35$  in chess reduces to  $b = \sqrt{35} \approx 6$  !!!

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## Key Points

- Game playing: what are the types of games?
- Minimax: definition, and how to get minmax values
- Minimax: evaluation
- $\alpha$ - $\beta$  pruning: why it saves time

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## $\alpha - \beta$ Pruning: Initialization

Along the path from the beginning to the current **state**:

- $\alpha$ : best MAX value
  - initialize to  $-\infty$
- $\beta$ : best MIN value
  - initialize to  $\infty$

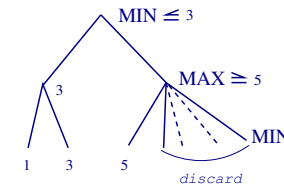
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## Overview

- formal  $\alpha - \beta$  pruning algorithm
- $\alpha - \beta$  pruning properties
- games with an element of chance
- state-of-the-art game playing with AI
- more complex games

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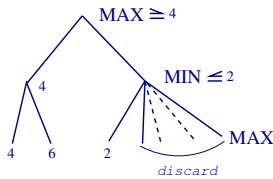
## $\alpha - \beta$ Pruning Algorithm: Max-Value



```
function Max-Value (state, game,  $\alpha$ ,  $\beta$ ) return utility value
 $\alpha$ : best MAX on path to state ;  $\beta$ : best MIN on path to state
if Cutoff(state) then return Utility(state)
 $v \leftarrow -\infty$ 
for each s in Successor(state) do
  ·  $v \leftarrow \text{Max}(\alpha, \text{Min-Value}(s, \text{game}, \alpha, \beta))$ 
  · if  $v \geq \beta$  then return v /* CUT!! */
  ·  $\alpha \leftarrow \text{Max}(\alpha, v)$ 
end
return v
```

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## $\alpha - \beta$ Pruning Algorithm: Min-Value



```

function Min-Value (state, game,  $\alpha$ ,  $\beta$ ) return utility value
 $\alpha$ : best MAX on path to state ;  $\beta$ : best MIN on path to state
if Cutoff(state) then return Utility (state)
 $v \leftarrow \infty$ 
for each s in Successor(state) do
  ·  $v \leftarrow \text{Min}(\beta, \text{Max-Value}(s, \text{game}, \alpha, \beta))$ 
  · if  $v \leq \alpha$  then return v /* CUT!! */
  ·  $\beta \leftarrow \text{Min}(\beta, v)$ 
end
return v
    
```

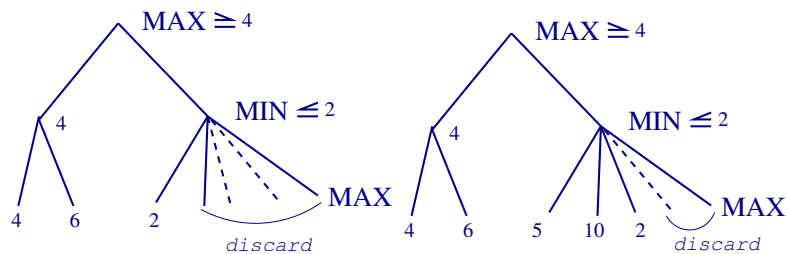
101

## $\alpha - \beta$ Pruning Tips

- At a MAX node:
  - Only  $\alpha$  is updated with the MAX of successors.
  - Cut is done by checking if returned  $v \geq \beta$ .
  - If all fails, MAX( $v$  of successors) is returned.
- At a MIN node:
  - Only  $\beta$  is updated with the MIN of successors.
  - Cut is done by checking if returned  $v \leq \alpha$ .
  - If all fails, MIN( $v$  of successors) is returned.

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## Ordering is Important for Good Pruning



- For MIN, sorting successor's utility in an **increasing** order is better (shown above; left).
- For MAX, sorting in **decreasing** order is better.

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## Games With an Element of Chance

Rolling the dice, shuffling the deck of card and drawing, etc.

- **chance nodes** need to be included in the minimax tree
- try to make a move that maximizes the **expected value**  $\rightarrow$  **expectimax**
- expected value of random variable  $X$ :

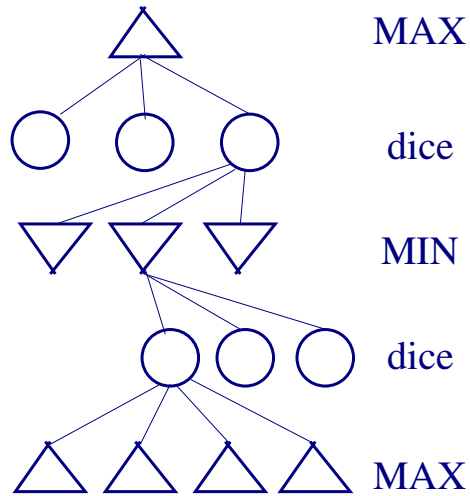
$$E(X) = \sum_x xP(x)$$

- expectimax

$$\text{expectimax}(C) = \sum_i P(d_i) \max_{s \in S(C, d_i)} (\text{utility}(s))$$

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## Game Tree With Chance Element



- chance element forms a new ply (e.g. dice, shown above)

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## Design Considerations for Probabilistic Games

- the **value** of evaluation function, not just the **scale** matters now! (think of what expected value is)
- time complexity:  $b^m n^m$ , where  $n$  is the number of distinct dice rolls
- pruning can be done if we are careful

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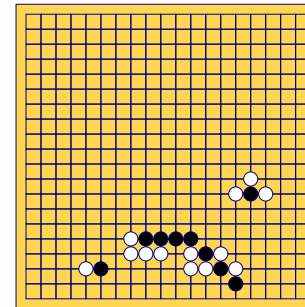
## State of the Art in Gaming With AI

- Chess: IBM's Deep Blue defeated Garry Kasparov (1997)
- Backgammon: Tesauro's Neural Network → top three (1992)
- Othello: smaller search space → superhuman performance
- Checkers: Samuel's Checker Program running on 10Kbyte (1952)

Genetic algorithms can perform very well on select domains.

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## Hard Games



The game of Go, popular in East Asia:

- $19 \times 19 = 361$  grid: branching factor is huge!
- search methods inevitably fail: need more structured rules
- the bet was high: \$1,400,000 prize for the first computer program to beat a select, 12-year old player. The late Mr. Ing Chang Ki (photo above) put up the money from his personal funds.

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## Key Points

- formal  $\alpha - \beta$  pruning algorithm: know how to apply pruning
- $\alpha - \beta$  pruning properties: evaluation
- games with an element of chance: what are the added elements?  
how does the minmax tree get augmented?