

CHAPTER 6

Reliability Modelling in Non-Markovian Systems

Introduction

Most reliability models assume that the up and down times of the components are exponentially distributed. This assumption leads to a Markovian model with constant interstate transition rates. The analysis in such cases is relatively simple and the numerical results can be easily obtained. The assumption is often valid for the up time but the down times are likely to have a non-exponential distribution. When the components are independent, the steady state results, as is shown later, are not affected by the shape of the distribution. In the case of dependent failures there can be a very definite affect.

If the distributions cannot be represented by a single exponential form then the process becomes non-Markovian and different techniques are required for process solution. This chapter presents some different methods for solving non-Markovian systems by application to specific models.

The Difficulty with Non-Markovian Processes

The essential difficulty with non-Markovian processes can be illustrated by the analysis of the reliability model of a binary unit (See Fig. 2.8). The up and down time durations are assumed to have the distribution $\lambda \exp\{-\lambda x\}$ and $f(y)$ respectively. Denoting the state occupied at time t by $Z(t)$, the equation of state 0, i.e. the up state can be written as

$$P\{Z(t + \Delta t) = 0\} = P\{Z(t + \Delta t) = 0 | Z(t) = 0\} P\{Z(t) = 0\} + P\{Z(t + \Delta t) = 0 | Z(t) = 1\} P\{Z(t) = 1\} \quad (6.1)$$

It can be seen that as $\Delta t \rightarrow 0+$,

$$P\{Z(t + \Delta t) = 0 | Z(t) = 0\} = 1 - \lambda \Delta t$$

The distribution of down time is, however, not exponential and therefore the repair rate $\mu(y)$ depends on the time y which the component has spent in state 1. The coefficient of the second term on the right hand side of Equation (6.1) under the condition that the component has been in state 1 for time y , is therefore $\mu(y)\Delta t$. The required coefficient can be obtained by integrating this conditional

coefficient over the distribution of the time spent in state 1 up to time t . It is this dependence of the transfer rate on the random duration y in the state 1 that is the essential difficulty in formulating Equation (6.1). If, however, the transition rates were to depend on time t in an explicitly known manner, there would not be any special difficulty. For example, if the repair rate were known as a function of time t , the coefficient of the second term would be $\mu(t)\Delta t$. The basic approach to avoiding this difficulty is to convert the non-Markov process into a Markov process by redefinition of the state space.

Method of Supplementary Variables

This is probably the most direct method of dealing with non-Markovian systems and will be illustrated by application to a bank of three single phase transformers with one spare. The state transition diagram with unrestricted repair and exponentially distributed down times has been presented in Fig. 4.4. The state

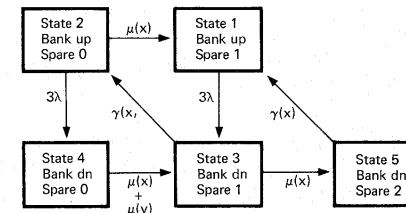


Fig. 6.1. The state transition diagram of a bank of three single phase transformers with one spare, unrestricted repair.

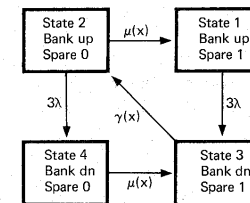


Fig. 6.2. The state transition diagram of a bank of three single phase transformers with one spare, restricted repair.

space diagram with arbitrary down time distributions and unrestricted repair is shown in Fig. 6.1. The corresponding model when the repair facilities are restricted is shown in Fig. 6.2. In this model, the repair on the second unit is not started until the repair on the failed unit is completed and it is reinstalled. The difference between the two state transition diagrams lies in the elimination of state 5 and the transition rate out of state 4. When the repair is unrestricted, both the transformers are simultaneously under repair, one for time x and the other for time y . With the repair restricted, only one transformer is under repair in state 4 and the other failed transformer is waiting. The repair on the second transformer cannot start until the first one has been installed and therefore state 5 is eliminated. The following notation has been used.

$f_u(x), f_r(x), f_c(x)$ = The probability density functions for the up time, repair duration and the change out period respectively

$S_u(x), S_r(x), S_c(x)$ = The survivor functions for the up time, repair duration and the change out period respectively

For example, $S_u(x) = P(U > x) = \int_x^\infty f_u(y) dy$, U is a random variable denoting the up time of the transformer.

Let Y be a continuous positive random variable specifying the duration of a particular state of the component until the termination of that state. Then the age specific transition rate is defined as

$$\phi(y) = \lim_{\Delta y \rightarrow 0^+} \frac{P(y < Y \leq y + \Delta y | y < Y)}{\Delta y} = \frac{f(y)}{S(y)}$$

$\mu(x)$ = The repair rate when the repair has been going on for x period of time

$\gamma(x)$ = The change out rate when the change out has been in operation for time x .

and

λ = The failure rate of the transformer. This is constant as the distribution for the up time is assumed to be exponential.

Exponential Models

If the repair and change out times are also assumed to be exponentially distributed, then the transition rates are all constant, i.e. $\mu(x) = \mu$ and $\gamma(x) = \gamma$. The process is Markovian as the random variables that generate it are all exponentially distributed.

Unrestricted Repair

This case has already been discussed in Chapter 4.

Restricted Repair

The equations for the state transition diagram of Fig. 6.2 can be written as

$$3\lambda p_1 - \mu p_2 = 0$$

$$(3\lambda + \mu)p_2 - \gamma p_3 = 0$$

$$\mu p_4 - 3\lambda p_2 = 0$$

$$\gamma p_3 - 3\lambda p_1 - \mu p_4 = 0$$

Any three equations of the above set together with the relationship

$$\sum_{i=1}^4 p_i = 1$$

can be solved to obtain the steady state availabilities

$$p_1 = 1/Z \quad p_2 = \left(\frac{3\lambda}{\mu}\right)/Z$$

$$p_3 = \left(\frac{3\lambda + \mu}{\gamma}\right)\frac{3\lambda}{\mu}/Z \quad p_4 = \left(\frac{3\lambda}{\mu}\right)^2/Z$$

where

$$Z = 1 + \frac{3\lambda}{\mu} \left(1 + \frac{3\lambda + \mu}{\gamma} + \frac{3\lambda}{\mu}\right)$$

$$P_{DN} = \frac{3\lambda}{\mu} \left(\frac{3\lambda}{\mu} + \frac{3\lambda + \mu}{\gamma}\right)/Z$$

and

$$f_{DN} = f_{UP} = 3\lambda \left(1 + \frac{3\lambda}{\mu}\right)/Z$$

Non-exponential Models

A. General Distributions for Repair and Change Out Periods, Restricted Repair.

When the repair and change out period distributions are non-exponential, the repair rate, $\mu(x)$ and the change out rate $\gamma(z)$ are functions of the age of the repair and change out. The stochastic process is, therefore, non-Markovian. The

most direct method of tackling this process is by the inclusion of sufficient supplementary variables in the specification of the state of the system to make the process Markovian. In the case of transformer banks the supplementary variables are the times expended in the repair or change out process. The resulting Markov process is in continuous time and has a state space which is multi-dimensional and partly discrete and partly continuous.

Define $p_i(t)$ = The probability that the system is in state i at time t .

$$p_i(x; t) = \lim_{\Delta x \rightarrow 0^+} \frac{P[\text{The system is in state } i, \text{ the current operation having been started in } (t-x-\Delta x, t-x)]}{\Delta x}$$

The current operation is the process of repair or change out due to which the system is in state i and as soon as this operation is terminated the system will transit out of this state. For example in Fig. 6.2

$p_4(x; t) \cdot \Delta x$ = The probability that the system is in state 4 at time t and the elapsed time since the repair started on the transformer bank lies in the interval $(x, x+\Delta x)$

The forward equations of the resulting Markov process can be written by considering the transitions during the increment Δt .

$$p_1(t + \Delta t) = p_1(t)(1 - 3\lambda\Delta t) + \Delta t \int_0^\infty p_2(x; t)\mu(x)dx$$

$$p_2(x + \Delta t; t + \Delta t) = p_2(x; t) \{1 - (\mu(x) + 3\lambda)\Delta t\}$$

$$p_3(x + \Delta t; t + \Delta t) = \{1 - \gamma(x)\Delta t\} p_3(x; t)$$

$$p_4(x + \Delta t; t + \Delta t) = \{1 - \mu(x)\Delta t\} p_4(x; t) + 3\lambda\Delta t p_2(x; t)$$

The resulting differential equations as $\Delta t \rightarrow 0^+$ are

$$\frac{\partial p_1(t)}{\partial t} = -3\lambda p_1(t) + \int_0^\infty p_2(x; t)\mu(x)dx \quad (6.2)$$

$$\frac{\partial p_2(x; t)}{\partial t} + \frac{\partial p_2(x; t)}{\partial x} = -\{3\lambda + \mu(x)\} p_2(x; t) \quad (6.3)$$

$$\frac{\partial p_3(x; t)}{\partial t} + \frac{\partial p_3(x; t)}{\partial x} = -\gamma(x) p_3(x; t) \quad (6.4)$$

$$\frac{\partial p_4(x; t)}{\partial t} + \frac{\partial p_4(x; t)}{\partial x} = -\mu(x) p_4(x; t) + 3\lambda p_2(x; t) \quad (6.5)$$

Equations (6.2)–(6.5) can be solved under the boundary conditions

$$p_2(0; t) = \int_0^\infty p_3(x; t)\gamma(x)dx \quad (6.6)$$

$$p_3(0; t) = 3\lambda p_1(t) + \int_0^\infty p_4(x; t)\mu(x)dx \quad (6.7)$$

and

$$p_4(0; t) = 0 \quad (6.8)$$

The boundary Equation (6.6) results from the fact that as soon as the spare or repaired transformer is reinstalled the system enters state 2. Similar reasoning holds for Equation (6.7). Equation (6.8) states that it is impossible to be in state 4 without the transformer being under repair for some time.

It is interesting to indicate at this point that Equations (6.2) – (6.8) can also be obtained using the frequency balancing concept outlined in Chapter 3. Following the approach given in this chapter, the equation for state 2 under the condition that repair has been in progress for time x can be written as

$$\begin{aligned} \Delta \{p_2(x; t)\Delta x\} &= -\{3\lambda + \mu(x)\} p_2(x; t)\Delta x \cdot \Delta t \\ \text{i.e.} \quad \Delta p_2(x; t) &= -\{3\lambda + \mu(x)\} p_2(x; t)\Delta t \end{aligned}$$

Now knowing that small increases in both x and t are Δt

$$\begin{aligned} \Delta p_2(x; t) &= \frac{\partial p_2(x; t)}{\partial t} \Delta t + \frac{\partial p_2(x; t)}{\partial x} \Delta x \\ &= -\{3\lambda + \mu(x)\} p_2(x; t)\Delta t \end{aligned}$$

That is

$$\frac{\partial p_2(x; t)}{\partial t} + \frac{\partial p_2(x; t)}{\partial x} = -\{3\lambda + \mu(x)\} p_2(x; t)$$

which is the same as Equation (6.3).

Since the primary interest is in the steady state availabilities, Equations (6.2)–(6.8) reduce to the equilibrium equations as $t \rightarrow \infty$

$$3\lambda p_1 = \int_0^\infty p_2(x)\mu(x)dx \quad (6.9)$$

$$\frac{\partial p_2(x)}{\partial x} = -\{3\lambda + \mu(x)\} p_2(x) \quad (6.10)$$

$$\frac{\partial p_3(x)}{\partial x} = -\gamma(x) p_3(x) \quad (6.11)$$

$$\frac{\partial p_4(x)}{\partial x} = -\mu(x)p_4(x) + 3\lambda p_2(x) \quad (6.12)$$

$$p_2(0) = \int_0^{\infty} p_3(x)\gamma(x)dx \quad (6.13)$$

$$p_3(0) = 3\lambda p_1 + \int_0^{\infty} p_4(x)\mu(x)dx \quad (6.14)$$

and
$$p_4(0) = 0 \quad (6.15)$$

In these equations

p_i = The steady state probability of being in state i

and

$p_i(x)\Delta x$ = The steady state probability of being in state i and the elapsed time since the current operation started lies in the interval $(x, x + \Delta x)$.

It should be noted that $p_i(x)\Delta x$ denotes the probability of a continuous state since x may lie anywhere in the interval $(0, \infty)$. The availability of the system condition denoted by state i can, therefore, be obtained by

$$p_i = \int_0^{\infty} p_i(x)dx \quad (6.16)$$

Equations (6.9)–(6.15) together with the normalizing equation

$$\sum_{i=1}^4 p_i = 1 \quad (6.17)$$

can be solved to obtain the steady state availabilities.

Equation (6.11) on solution gives

$$p_3(x) = p_3(0) \exp\left\{-\int_0^x \gamma(w)dw\right\} \quad (6.18)$$

Since it is well known that

$$S_c(x) = \exp\left\{-\int_0^x \gamma(w)dw\right\}$$

$$p_3 = \int_0^{\infty} p_3(x)dx = p_3(0) \int_0^{\infty} S_c(x)dx = p_3(0)/\gamma$$

where

$$\frac{1}{\gamma} = \text{the mean change out period}$$

Therefore

$$p_3 = \frac{1}{\gamma} \left\{ 3\lambda p_1 + \int_0^{\infty} p_4(x)\mu(x)dx \right\} \quad (6.19)$$

Solving Equation (6.10)

$$p_2(x) = p_2(0) \exp\left\{-\int_0^x (3\lambda + \mu(w))dw\right\} \quad (6.20)$$

$$p_2 = \int_0^{\infty} p_2(x)dx = p_2(0) \int_0^{\infty} \exp(-3\lambda x) S_r(x)dx$$

$$= \frac{1}{3\lambda} (1-E) \int_0^{\infty} p_3(x)\gamma(x)dx \quad (6.21)$$

where

$$E = \int_0^{\infty} f_r(x) e^{-3\lambda x} dx = E_r(e^{-3\lambda x}) \quad (6.22)$$

Substituting from Equation (6.18) into (6.20)

$$p_2 = \frac{1}{3\lambda} (1-E) p_3 \gamma \quad (6.23)$$

Substituting (6.20) into (6.9)

$$3\lambda p_1 = p_2(0) \int_0^{\infty} \exp\left\{-\int_0^x (3\lambda + \mu(w))dw\right\} \mu(x)dx$$

$$= p_2(0) \int_0^{\infty} f_r(x) e^{-3\lambda x} dx = \gamma E p_3 \quad (6.24)$$

Equation (6.12) can be solved using the boundary condition (6.15) giving

$$p_4(x) = \exp\left\{-\int_0^x \mu(w)dw\right\} \int_0^x 3\lambda p_2(y) \exp\left\{\int_0^y \mu(v)dv\right\} dy$$

Substituting from Equation (6.20) and simplifying

$$p_4(x) = \gamma p_3 (1 - e^{-3\lambda x}) S_r(x) \quad (6.25)$$

and

$$p_4 = \int_0^{\infty} p_4(x)dx = \gamma p_3 \left[\frac{1}{\mu} - \frac{(1-E)}{3\lambda} \right] \quad (6.26)$$

where

$$\frac{1}{\mu} = \text{Mean repair time}$$

From (6.24)

$$p_3 = \frac{3\lambda}{\gamma E} p_1 \quad (6.27)$$

From (6.23)

$$p_2 = \frac{1-E}{3\lambda} \gamma p_3 = \frac{1-E}{E} p_1 \quad (6.28)$$

From (6.26) and (6.27)

$$p_4 = \frac{3\lambda}{E} \left[\frac{1}{\mu} - \frac{1}{3\lambda} (1-E) \right] p_1 \quad (6.29)$$

Substituting from Equations (6.27)–(6.29) into (6.17) and simplifying

$$\begin{aligned} p_1 &= 1/D \quad \text{where} \quad D = 1 + \frac{3\lambda}{E} \left(\frac{1}{\mu} + \frac{1}{\gamma} \right) \\ p_2 &= \frac{1-E}{DE} \quad p_3 = \frac{3\lambda}{\gamma ED} \quad \text{and} \quad p_4 = \frac{3\lambda}{DE} \left[\frac{1}{\mu} - \frac{1-E}{3\lambda} \right] \\ p_{DN} &= p_3 + p_4 = \frac{3\lambda}{DE} \left[\frac{1}{\gamma} + \frac{1}{\mu} - \frac{1-E}{3\lambda} \right] \end{aligned} \quad (6.30)$$

The frequency

$$\begin{aligned} f_{DN} &= \int_0^{\infty} p_3(x) \gamma(x) dx = p_3(0) \int_0^{\infty} f_c(x) dx \\ &= p_3(0) = \gamma p_3 = \frac{3\lambda}{DE} \\ f_{UP} &= 3\lambda(p_1 + p_2) = \frac{3\lambda}{DE} = f_{DN} \end{aligned} \quad (6.31)$$

It can be seen that the equations using a general distribution are different from those derived assuming an exponential distribution. The availabilities are now dependent upon the transform E which for the exponential distribution of repair reduces to

$$= \int_0^{\infty} \mu e^{-(3\lambda+\mu)x} dx = \frac{\mu}{3\lambda + \mu}$$

It is interesting to note that the steady state probabilities depend only upon the reciprocal of the mean change out time, i.e. the average change out rate. Therefore as long as the mean change out time stays the same, the form of its probability density function does not affect the steady state probabilities and frequencies (see that $f_{DN} = p_3 \gamma$). This always happens when the operation is started and completed in the same state as for example in the present case the reinstallation is started and completed in state 3 of the system. If, however, the reinstallation were not always completed in state 3, then the form of the probability density affects the steady state probabilities. This is seen in the next section.

B. General Distribution for Change Out, Exponential Distribution for Repair, Unrestricted Repair.

It can be seen from Fig. 6.1 that in this case the reinstallation phase initiated in state 3 may not always be completed in state 3 but sometimes in state 5. The concept of average change out rate thus does not apply here.

Defining

$$p_i(x; t) \Delta x = \text{The probability that the system is in the state } i \text{ at time } t \text{ and the elapsed time since the start of reinstallation lies in the interval } (x, x + \Delta x)$$

the differential equations can be written as

$$\frac{\partial p_1(t)}{\partial t} = -3\lambda p_1(t) + \mu p_2(t) + \int_0^{\infty} p_5(x; t) \gamma(x) dx \quad (6.32)$$

$$\frac{\partial p_2(t)}{\partial t} = -(3\lambda + \mu) p_2(t) + \int_0^{\infty} p_3(x; t) \gamma(x) dx \quad (6.33)$$

$$\frac{\partial p_3(x; t)}{\partial t} + \frac{\partial p_3(x; t)}{\partial x} = -\{\mu + \gamma(x)\} p_3(x; t) \quad (6.34)$$

$$\frac{\partial p_4(t)}{\partial t} = -2\mu p_4(t) + 3\lambda p_2(t) \quad (6.35)$$

and

$$\frac{\partial p_5(x; t)}{\partial t} + \frac{\partial p_5(x; t)}{\partial x} = -\gamma(x) p_5(x; t) + \mu p_3(x; t) \quad (6.36)$$

These equations can be solved using the boundary conditions

$$p_3(0; t) = 3\lambda p_1(t) + 2\mu p_4(t) \quad (6.37)$$

and

$$p_5(0; t) = 0 \quad (6.38)$$

Equation (6.38) indicates that the reinstallation phase is never started in state 5.

Under equilibrium conditions, i.e. as $t \rightarrow \infty$, the Equations (6.32)–(6.38) reduce to

$$3\lambda p_1 = \mu p_2 + \int_0^\infty p_5(x)\gamma(x)dx \quad (6.39)$$

$$(3\lambda + \mu)p_2 = \int_0^\infty \gamma(x)p_3(x)dx \quad (6.40)$$

$$\frac{\partial p_3(x)}{\partial x} = -\{\mu + \gamma(x)\}p_3(x) \quad (6.41)$$

$$2\mu p_4 = 3\lambda p_2 \quad (6.42)$$

$$\frac{\partial p_5(x)}{\partial x} = -\gamma(x)p_5(x) + \mu p_3(x) \quad (6.43)$$

$$p_3(0) = 3\lambda p_1 + 2\mu p_4 \quad (6.44)$$

and

$$p_5(0) = 0 \quad (6.45)$$

On solving Equation (6.41) and substituting from Equation (6.44)

$$p_3(x) = (3\lambda p_1 + 2\mu p_4) \exp\left\{-\int_0^x (\mu + \gamma(w))dw\right\} \quad (6.46)$$

Therefore

$$\begin{aligned} p_3 &= \int_0^\infty p_3(x)dx = (3\lambda p_1 + 2\mu p_4) \int_0^\infty e^{-\mu x} S_c(x)dx \\ &= (3\lambda p_1 + 2\mu p_4) \left(\frac{1 - E_c}{\mu}\right) \end{aligned} \quad (6.47)$$

where

$$E_c = \int_0^\infty f_c(x) e^{-\mu x} dx$$

Substituting (6.46) and (6.40) and simplifying

$$(3\lambda + \mu)p_2 = (3\lambda p_1 + 2\mu p_4)E_c \quad (6.48)$$

Solving Equations (6.43) using (6.45)

$$p_5(x) = \exp\left\{-\int_0^x \gamma(w)dw\right\} \int_0^x p_3(y) \exp\left\{\int_0^y \gamma(v)dv\right\} dy$$

Substituting from Equation (6.46) and simplifying

$$p_5(x) = p_3(0)(1 - e^{-\mu x})S_c(x) \quad (6.49)$$

Therefore

$$p_5 = \int_0^\infty p_5(x)dx = (3\lambda p_1 + 2\mu p_4) \left[\frac{1}{\gamma} - \frac{1}{\mu} + \frac{E_c}{\mu}\right] \quad (6.50)$$

Substituting from (6.42) into (6.48) and simplifying

$$p_2 = \frac{3\lambda E_c}{3\lambda(1 - E_c) + \mu} p_1 \quad (6.51)$$

From (6.42)

$$p_4 = \frac{3\lambda}{2\mu} p_2 = \frac{9\lambda^2 E_c}{2\mu[3\lambda(1 - E_c) + \mu]} p_1 \quad (6.52)$$

Substituting for p_4 in (6.47)

$$p_3 = \frac{3\lambda}{\mu} (1 - E_c) \left[1 + \frac{3\lambda E_c}{3\lambda(1 - E_c) + \mu}\right] p_1 \quad (6.53)$$

Substituting for p_4 in (6.50)

$$p_5 = \frac{3\lambda}{\mu} \left(\frac{\mu}{\gamma} - 1 + E_c\right) \left[1 + \frac{3\lambda E_c}{3\lambda(1 - E_c) + \mu}\right] p_1 \quad (6.54)$$

Substituting into the normalizing equation

$$\sum_{i=1}^5 p_i = 1$$

and simplifying

$$p_1 = \frac{2\gamma\mu\{3\lambda(1 - E_c) + \mu\}}{B}$$

where

$$B = \{3\lambda(1 - E_c) + \mu\}(2\gamma\mu + 6\lambda\mu) + 3\lambda E_c(2\gamma\mu + 6\lambda\mu + 3\lambda\gamma)$$

$$p_2 = \frac{6\lambda\gamma\mu}{B} E_c \quad p_3 = \frac{6\lambda\gamma(3\lambda + \mu)}{B} (1 - E_c)$$

$$p_4 = \frac{9\lambda^2\gamma}{B} E_c \quad p_5 = \frac{6\lambda(3\lambda + \mu)(\mu - \gamma + \gamma E_c)}{B}$$

$$p_{DN} = p_3 + p_4 + p_5 = \frac{9\lambda^2\gamma E_c + 6\lambda\mu(3\lambda + \mu)}{B} \quad (6.55)$$

$$p_{UP} = \frac{6\lambda\gamma\mu + 2\gamma\mu^2}{B} \quad (6.56)$$

The Frequency $f_{DN} = \int_0^\infty p_3(x)\gamma(x)dx + \int_0^\infty p_5(x)\gamma(x)dx$

$$\begin{aligned}
 &= p_3(0) \int_0^\infty \gamma(x) \exp \left\{ - \int_0^x (\mu + \gamma(w)) dw \right\} dx \\
 &\quad + p_3(0) \int_0^\infty (1 - e^{-\mu x}) f_c(x) dx \\
 &= p_3(0) = 3\lambda p_1 + 2\mu p_4 = 3\lambda(p_1 + p_2) = 3\lambda p_{UP} = f_{UP} \quad (6.57)
 \end{aligned}$$

Semi-Markov Processes

A semi-Markov process is a stochastic process in which the transitions from state to state are in accordance with a Markov Chain but the time spent in a state before a transition occurs is random.

Consider a stochastic process $Z(t)$ which at time t can be in any of the n distinct states, $Z(t) = i$ denoting that the stochastic process is in state i at time t . Let the time just after the m th transition be denoted by t_m . The stochastic process is Markovian if

$$\begin{aligned}
 P[Z(t_m) = j | Z(t_{m-1}) = i, Z(t_{m-2}) = k, \dots, Z(t_1) = l] \\
 = P[Z(t_m) = j | Z(t_{m-1}) = i]
 \end{aligned}$$

If this probability is independent of the number of the transition, the process is time homogeneous.

Let $\alpha_{ij} = P[Z(t_m) = j | Z(t_{m-1}) = i]$

be the probability of going from state i to j in one step. The matrix of these transition probabilities will be denoted by $A = (\alpha_{ij})$. Given that the state i has just been entered, the probability of going to state j is specified by α_{ij} and $F_{ij}(\cdot)$ is the distribution function of the waiting time X_{ij} in state i given that the next transition will be to state j . The transitions can therefore be thought of as taking place in two stages. When the process has just entered state i , the next state j is selected according to the matrix A but once j has been picked, the waiting time X_{ij} is specified by $F_{ij}(\cdot)$. The Markov Chain, $A = (\alpha_{ij})$ associated with the semi-Markov process is called an embedded Markov Chain.

Discrete time and continuous time Markov chains are special cases of the semi-Markov process. For a discrete time Markov chain

$$F_{ij}(t) = \begin{cases} 0 & \text{for } t < c \\ 1 & \text{for } t \geq c \end{cases}$$

where c is a constant. The discrete time Markov chain is therefore a semi-Markov process in which the waiting time X_{ij} is constant. For the continuous time Markov chain

$$F_{ij}(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 - \exp \{-\lambda_i t\} & \text{for } t > 0, \lambda_i > 0 \end{cases}$$

where λ_i is the reciprocal of the mean time of X_{ij} .

Let $Q_{ij}(t) = \alpha_{ij} F_{ij}(t)$

It should be noted that $F_{ij}(t)$ represents the conditional probability that a transition will take place in time $\leq t$ given that the process has just entered i and will next enter j . $Q_{ij}(t)$ on the other hand is the probability that given that the process has just entered i , it will transit to state j in time less than or equal to t . Let

$$p_{ij}(t) = P[Z(t) = j | Z(0) = i]$$

Then

$$p_{ij}(t) = \sum_{k=0}^n \int_0^t p_{kj}(t-x) dQ_{ik}(x) \quad (6.58)$$

and

$$p_{ij}(t) = 1 - \sum_{k=0}^n \int_0^t (1 - p_{kj}(t-x)) dQ_{ik}(x) \quad (6.59)$$

The above equations involve convolution integrals of the form

$$\int_0^t g(t-x) f(x) dx$$

The Laplace of this integral is of the simple form $\bar{g}(s)\bar{f}(s)$. These integral equations can therefore be reduced to linear equations by taking Laplace transforms. The Laplace transform of Equations (6.58) and (6.59) in the matrix form can be written as

$$s\bar{P}(s) = [I - \bar{G}(s)]^{-1} [I - \bar{F}_d(s)] \quad (6.60)$$

where

$\bar{P}(s)$ is the matrix whose $\bar{p}_{ij}(s)$ term is the Laplace of the probability $p_{ij}(t)$
 $\bar{G}(s)$ is the matrix whose ij^{th} term is $\alpha_{ij} \bar{f}_{ij}(s)$
 $\bar{F}_d(s)$ is the diagonal matrix whose ii^{th} term is equal to $\sum_k \bar{f}_{ik}(s) \alpha_{ik}$
 I is the identity matrix.

The probabilities can be obtained in the Laplace form and the time specific solutions found by inversion. More often, however, the steady state probabilities are required and these can be obtained directly using the following relationship

$$p_i = \frac{m_i \Pi_i}{\sum_j m_j \Pi_j} \quad (6.61)$$

where

π_i = The steady state probability that the embedded Markov chain is in state i .
 m_i = The mean residence time in state i .

The steady state probability vector can be found by solving

$$\pi A = \pi$$

along with

$$\sum_i \pi_i = 1$$

The application of a semi-Markov process is illustrated with an example of two three phase transformers in parallel. When a fault develops on either of the transformers, both the transformers are shut down. After the defective transformer has been isolated, the good one is returned for operation. The state transition diagram is shown in Fig. 6.3.

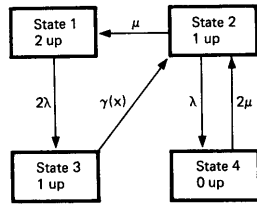


Fig. 6.3 The state transition diagram for two three phase transformers in parallel

The up time and repair time are assumed to be exponentially distributed but the change out time is assumed to have an arbitrary probability density function $f_c(t)$ having a Laplace transform $\bar{f}_c(s)$. The A matrix of transformation probabilities is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{\mu}{\mu + \lambda} & 0 & 0 & \frac{\lambda}{\lambda + \mu} \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The steady state probabilities of the Markov Chain are

$$\pi_1 = \frac{1}{3 + 2\lambda/\mu} \quad \pi_2 = \left(\frac{\mu + \lambda}{\mu}\right) \left/ \left(3 + \frac{2\lambda}{\mu}\right)\right.$$

$$\pi_3 = \frac{1}{3 + 2\lambda/\mu} \quad \pi_4 = \left(\frac{\lambda}{\mu}\right) \left/ \left(3 + \frac{2\lambda}{\mu}\right)\right.$$

Also

$$m_1 = \frac{1}{2\lambda} \quad m_2 = \frac{1}{\mu + \lambda} \quad m_3 = \frac{1}{\gamma} \quad m_4 = \frac{1}{2\mu}$$

where

$$\frac{1}{\gamma} = \text{mean change out time.}$$

Substituting into Equation (6.61)

$$p_1 = \frac{1}{Z} \quad p_2 = \frac{2\lambda}{\mu Z}$$

$$p_3 = \frac{2\lambda}{\gamma Z} \quad p_4 = \frac{\lambda^2}{\mu^2 Z}$$

where

$$Z = 1 + \frac{\lambda^2}{\mu^2} + \frac{2\lambda}{\mu} + \frac{2\lambda}{\gamma}$$

It can be seen that the probability density function of the change out time $f_c(t)$ enters these expressions through the mean value $1/\gamma$. The same results would therefore be obtained if the change out time were assumed to be exponentially distributed with the transition rate equal to the reciprocal of the mean change out time. This is due to the fact that the change out operation starts and ends in the same state, number 3. It should, however, be noted that the time dependent solution would be different from that assuming an exponential distribution.

Device of Stages

The device of stages is a method of representing a non-exponentially distributed state by a combination of stages each of which is exponentially distributed. The method, therefore, represents a non-Markovian model by an equivalent Markovian model which is generally simpler to solve. Any distribution with a rational Laplace transform can, in principle, be represented exactly by a stage combination. Though this may involve complex probabilities associated with the fictitious stages, the probabilities of the actual states of the system are always real. Many distributions not necessarily having a rational Laplace transform can also be reasonably approximated by relatively simple stage combinations. The application of this technique involves the following steps.

1 Selection of a Stage Combination

When the distribution has a rational Laplace transform, the stage combination can be found by examining the roots of this transform. In other probability distribution cases or of directly fitted data cases a suitable guess has to be made. The probability density function and the hazard rate function of a given distribution or data should be examined. A number of simple stage combinations and their characteristics are described later in the chapter. The characteristics of the given distribution or data should be compared with those of the stage combinations, and a suitable combination should be selected. The difference between distributions like gamma, Weibul and lognormal become significant only in the tail regions, their hazard functions are, however, quite distinct. Therefore both the density function and the hazard function should be compared when selecting a proper stage combination.

2 Determination of Parameters

When a stage combination has been selected, the next step is the derivation of its parameters from those of the distribution. This can be done by a moment matching technique which is described in this chapter.

In addition to the constant hazard rate exhibited only by the exponential distribution, there are four basic hazard rate shapes.

1. Increasing hazard rate (Fig. 6.4a)
2. Decreasing hazard rate (Fig. 6.4b)
3. Initial period of decreasing hazard rate followed by increasing rate (Fig. 6.4c)
4. Initial period of increasing hazard rate followed by decreasing rate (Fig. 6.4d).

The combinations discussed in this chapter are capable of generating these shapes. An awareness of this characteristic can be very useful in selecting a proper combination.

General Technique for Deriving the Characteristics of Stage Combinations

The probability density function, the survivor function and the hazard rate function of a given stage combination can be derived in a number of ways but a general technique, often helpful in difficult situations is described below. Let O be the equivalent state of a given stage combination. The transitions from this state are assumed to be terminated in an absorbing state A , as shown in Fig. 6.5. The process is assumed to start in the same stage, into which it would first transit when state O is entered. The time spent in state O is identical with the time from the origin as no transition is made from A to O . Under these conditions

- $f_0(x)$ = The probability density function of state O , i.e., the stage combination
- = The time specific frequency of transiting from O to A

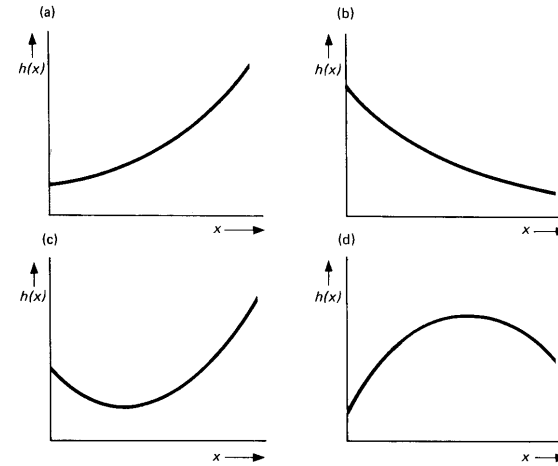


Fig. 6.4 Some typical hazard rate functions.

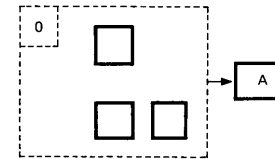


Fig. 6.5 State transition diagram to determine the characteristics of a stage combination

$$= \sum_{i \in 0} p_i(x) \lambda_{iA} \tag{6.62}$$

where

λ_{iA} = The transition rate from state i , member of 0 , to the absorbing state A .

$S_0(x)$ = The survivor function of the stage combination

= The probability of being in state 0 at any time

$$= \sum_{i \in 0} p_i(x) \quad (6.63)$$

and

$$\begin{aligned} \phi_0(x) &= \text{The hazard rate} \\ &= f_0(x)/S_0(x) \\ &= \sum_{i \in 0} p_i(x) \lambda_{iA} / \sum_{i \in 0} p_i(x) \\ &= \text{The equivalent transfer rate from state 0 to } A \end{aligned} \quad (6.64)$$

The evaluation of Equations (6.62) – (6.64) involves the calculation of the probabilities of being in various stages of state O . This is generally a simpler method. The main advantage is that though explicit formulae may not be possible, numerical values of the state probabilities can be easily determined using the techniques discussed in Chapter 3. The numerical values of the three characteristics can then be calculated using the above equations. The method will become more clear later when the specific equations for the stage combinations are derived.

Stage Combinations

We will now describe some specific stage combinations which are simple but versatile enough to approximate a variety of probability distribution functions. The stage combinations will be shown as approximating the down time distribution of a two state component whose up state is assumed to be exponentially distributed. This arrangement is only for the sake of illustration, the stage combination may be used to represent a state in any other context.

The Stages in Series

The stages are traversed in a sequential order and the non-negative continuous random variable X , representing the component state duration is the sum of ' a ' independent exponentially distributed random variables. The Laplace transform of the distribution of X can then easily be proved to be

$$\frac{\rho_1 \cdots \rho_a}{(\rho_1 + s) \cdots (\rho_a + s)} \quad (6.65)$$

The probability density function can be obtained by expressing Equation (6.64) in the partial fraction form

$$\sum \frac{A_i \rho_i}{\rho_i + s}$$

The probability density function, therefore, is

$$\sum A_i \rho_i \exp(-\rho_i x) \quad (6.66)$$

It should be noted that $\sum A_i = 1$ but A_i do not all lie in $(0, 1)$ and therefore $\{A_i\}$ is not a probability distribution. It can be easily shown that for a fixed a any fractional coefficient of variation between 1 and $1/\sqrt{a}$ can be produced by a suitable choice of $\{\rho_i\}$.

The positions of the poles of Equation (6.65) determine the transition probabilities and the number of poles equals the number of stages.

When all the stages are identically distributed with parameter ρ , Expression (6.65) reduces to

$$\left(\frac{\rho}{\rho + s} \right)^a \quad (6.67)$$

and the corresponding probability density function is the Special Erlangian distribution

$$\frac{\rho(\rho x)^{a-1} e^{-\rho x}}{(a-1)!} \quad (6.68)$$

where a is a positive integer. The corresponding survivor function is

$$e^{-\rho x} \sum_{i=1}^a \frac{(\rho x)^{i-1}}{(i-1)!}$$

The characteristics of a family of Special Erlangian Distributions with a constant mean of one day are shown in Fig. 6.6. The exponential is a special case of the Special Erlangian Distribution with $\rho = 1$.

A generalization of Equation (6.68) is to replace the parameter a restricted to integer values by a parameter having any real positive value. The probability density function of Equation (6.68) then becomes the Gamma distribution

$$\frac{\rho(\rho x)^{\alpha-1} e^{-\rho x}}{\Gamma(\alpha)} \quad (6.69)$$

where the gamma function

$$\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1} e^{-u} du$$

The mean and standard deviation of the gamma distribution are α/ρ and $\sqrt{\alpha}/\rho$ respectively and the fractional coefficient of variation is therefore $1/\sqrt{\alpha}$. For a fixed mean $\mu = \alpha/\rho$ any coefficient of variation between 1 and $1/\sqrt{\alpha}$ may be obtained by varying α and ρ in the same proportion. If μ is kept constant then as $\alpha, \rho \rightarrow \infty$, the coefficient of variation approaches zero,

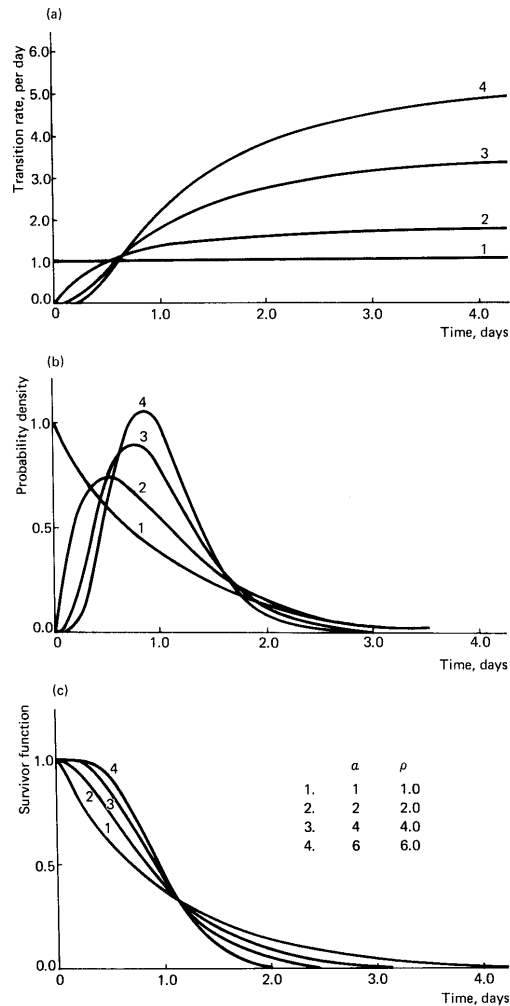


Fig. 6.6 Characteristics of the special Erlangian distribution.

i.e. there is no dispersion about the mean. This corresponds to the case of constant state duration of the component.

Many empirical distributions can be represented, at least approximately by a suitable choice of parameters α and ρ . It should, however, be realized that as α is not always an integer it may not be possible to interpret directly the distribution in terms of stages. When α is not an integer, it is preferable to solve for integer ' α ' using Equation (6.68) and then the numerical answer for α can be obtained by interpolation.

The Stages in Parallel

When the stages are in parallel, there is a probability distribution $\{\omega_i\}$, $i = 1, \dots, a$ such that the random variable X denoting the state duration of the component, has the probability ω_i of beginning on the i th stage, the life thereafter consisting of a single stage, i.e. only a single stage is used in any one realization of X . The probability density function of X is given by

$$\sum_{i=1}^a \omega_i \rho_i e^{-\rho_i x} \tag{6.70}$$

The probability density function of Equation (6.70) is formally equivalent to Equation (6.66) with the important difference that ω_i are all non-negative whereas there is no such restriction on A_i . The Laplace transform of Equation (6.70) is a rational one, i.e. the ratio of a polynomial of degree at the most ' a ' to a polynomial of the degree ' a '. If in practice, only two stages in parallel are required, the Expression (6.70) reduces to

$$\omega_1 \rho_1 e^{-\rho_1 x} + \omega_2 \rho_2 e^{-\rho_2 x} \tag{6.71}$$

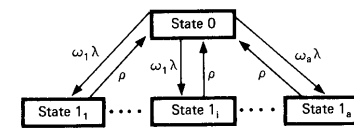


Fig. 6.7 The state transition diagram for a component having an exponentially distributed up time and whose down time has a probability density function of Equation (6.70)

It can be shown that by a suitable choice of ω_i, ρ_1, ρ_2 and the distribution of Equation (6.71) can have any desired mean and any fractional coefficient of variation between 1 and ∞ . The state transition diagram for a component whose

up time is exponentially distributed with parameter λ and whose down time has probability density function (6.70) is shown in Fig. 6.7. The total transition rate out of the up state, i.e. O is λ but it is directed towards different parallel stages of the down time according to the probability distribution $\{\omega_i\}$. That this transition diagram does represent the distribution can be seen by solving for $p_0(t)$. The differential equations are

$$p'_0(t) = -\lambda p_0(t) + \sum_{i=1}^a \rho_i p_{1i}(t) \tag{6.72}$$

$$p'_{1i}(t) = -\rho_i p_{1i}(t) + \omega_i \lambda p_0(t) \tag{6.73}$$

Taking the Laplace transform with the initial condition $p_0(0) = 1$

$$s\bar{p}_0(s) = 1 - \lambda\bar{p}_0(s) + \sum_{i=1}^a \rho_i \bar{p}_{1i}(s) \tag{6.74}$$

and

$$s\bar{p}_{1i}(s) = -\rho_i \bar{p}_{1i}(s) + \omega_i \lambda \bar{p}_0(s) \tag{6.75}$$

where $\bar{p}(s)$ is the Laplace of $p(t)$.

Substituting for $p_{1i}(s)$ from (6.75) into (6.74) and simplifying

$$\bar{p}_0(s) = \frac{1}{D(s)} = \frac{1}{s + \lambda - \lambda \sum_{i=1}^a \frac{\omega_i \rho_i}{s + \rho_i}} \tag{6.76}$$

It can be easily proved that for a component having an exponentially distributed up time and having a down time distribution of $f(t)$

$$\bar{p}_0(s) = \frac{1}{s + \lambda - \lambda \bar{f}(s)} \tag{6.77}$$

where $\bar{f}(s)$ = the Laplace transform of $f(t)$.

Substituting the Laplace of the probability density function (6.70)

$$\bar{p}_0(s) = \frac{1}{s + \lambda - \lambda \sum_{i=1}^a \frac{\omega_i \rho_i}{s + \rho_i}}$$

This expression is the same as Equation (6.76) derived using the state transition diagram of Fig. 6.7. The state diagram is, therefore, an accurate representation. A generalization of Equation (6.71) is to have two series stage combinations

in parallel as shown in Fig. 6.8. The expression for the probability density function is

$$\omega_1 \rho_1 e^{-\rho_1 x} \frac{(\rho_1 x)^{a_1-1}}{(a_1-1)!} + \omega_2 \rho_2 e^{-\rho_2 x} \frac{(\rho_2 x)^{a_2-1}}{(a_2-1)!} \tag{6.78}$$

The survivor function is

$$\omega_1 e^{-\rho_1 x} \sum_{i=1}^{a_1} \frac{(\rho_1 x)^{i-1}}{(i-1)!} + \omega_2 e^{-\rho_2 x} \sum_{i=1}^{a_2} \frac{(\rho_2 x)^{i-1}}{(i-1)!}$$

The mean and the variance are

$$\text{Mean} = \omega_1 a_1 / \rho_1 + \omega_2 a_2 / \rho_2$$

$$\text{Variance} = [\omega_1 a_1 (1 + a_1) / \rho_1^2 + \omega_2 a_2 (1 + a_2) / \rho_2^2] - [\omega_1 a_1 / \rho_1 + \omega_2 a_2 / \rho_2]^2$$

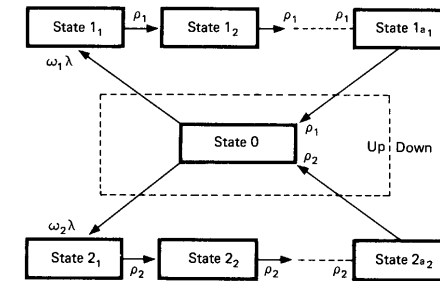


Fig. 6.8 The state transition diagram of a system with the down state represented by two series stages in parallel

The Expression (6.78) is a mixture of two Special Erlangian distributions and can approximate a wider range of distributions than the Special Erlangian. The combination has only five independent parameters. The various characteristics of this combination are shown in Fig. 6.9. These characteristics cover almost all the four types mentioned earlier. The theoretical analysis of the shape of the hazard rate function is given in Appendix II.

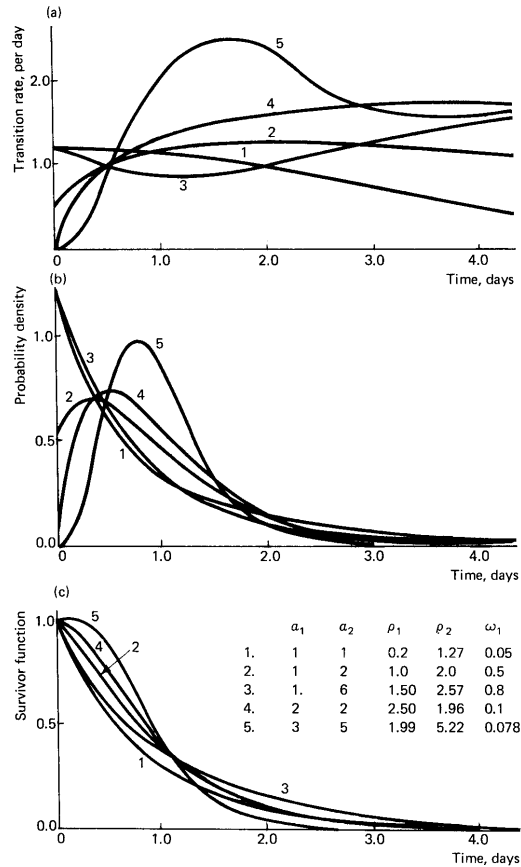


Fig. 6.9 Some characteristics of the mixture of two special Erlangian distributions

Series Stages in Series with a Distinctive Stage

The General Erlangian distribution (6.66) can generate a wider range of distribution than the Special Erlangian distribution but the number of parameters involved tends to be large. A special case combining the advantages of both is a series of identical stages in series with a distinctive stage as shown in Fig. 6.10. This model has three parameters:

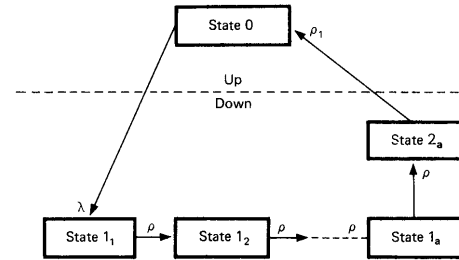


Fig. 6.10 The state transition diagram of a system with the down state represented by a number of series stages in series with a distinctive stage.

the transition rate ρ of each stage and the number of stages a in the identical series stages and the transition rate ρ_1 for the distinctive stage. The probability density function is

$$f(x) = \rho_1 \left(\frac{\rho}{\rho - \rho_1} \right)^a \left[e^{-\rho_1 x} - e^{-\rho x} \sum_{i=1}^a \frac{\{(\rho - \rho_1)x\}^{i-1}}{(i-1)!} \right] \tag{6.79}$$

The survivor function is

$$S(x) = e^{-\rho x} \sum_{i=1}^a \frac{(\rho x)^{i-1}}{(i-1)!} + \left(\frac{\rho}{\rho - \rho_1} \right)^a \times \left[e^{-\rho_1 x} - e^{-\rho x} \sum_{i=1}^a \frac{\{(\rho - \rho_1)x\}^{i-1}}{(i-1)!} \right] \tag{6.80}$$

and the hazard rate function

$$\phi(x) = \frac{f(x)}{S(x)}$$

The mean of this distribution is $a/\rho + 1/\rho_1$

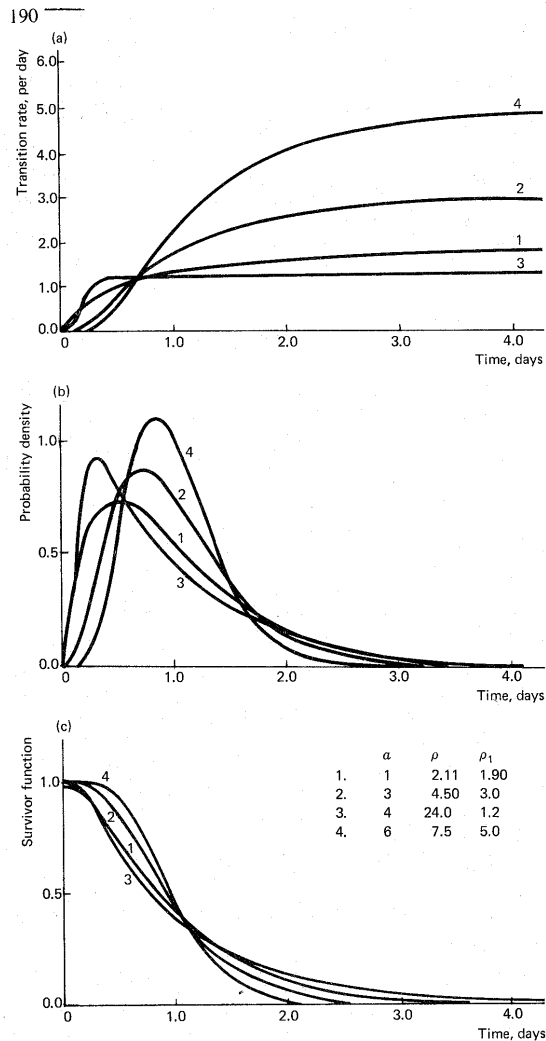


Fig. 6.11 Some characteristics of the distribution associated with series stages in series with a distinctive stage.

The characteristics of this distribution are illustrated in Fig. 6.11 and the theoretical analysis of the hazard rate is given in Appendix III.

Series Stages in Series with Two Parallel Stages

This combination has a series of identical stages followed by a stage with probability ω_1 or by another stage with probability ω_2 , as shown in Fig. 6.12a. It has five parameters and the expression for the probability density function is

$$f(x) = \omega_1 \rho_1 \left(\frac{\rho}{\rho - \rho_1} \right)^a \left[e^{-\rho_1 x} - e^{-\rho x} \sum_{i=1}^a \frac{\{(\rho - \rho_1)x\}^{i-1}}{(i-1)!} \right] + \omega_2 \rho_2 \left(\frac{\rho}{\rho - \rho_2} \right)^a \left[e^{-\rho_2 x} - e^{-\rho x} \sum_{i=1}^a \frac{\{(\rho - \rho_2)x\}^{i-1}}{(i-1)!} \right] \quad (6.81)$$

The survivor function can be expressed by

$$S(x) = \sum_{i=1}^a \frac{(\rho x)^{i-1}}{(i-1)!} e^{-\rho x} + \omega_1 \left(\frac{\rho}{\rho - \rho_1} \right)^a \times \left[e^{-\rho_1 x} - e^{-\rho x} \sum_{i=1}^a \frac{\{(\rho - \rho_1)x\}^{i-1}}{(i-1)!} \right] + \omega_2 \left(\frac{\rho}{\rho - \rho_2} \right)^a \left[e^{-\rho_2 x} - e^{-\rho x} \sum_{i=1}^a \frac{\{(\rho - \rho_2)x\}^{i-1}}{(i-1)!} \right] \quad (6.82)$$

The transition rate function can be found by

$$\phi(x) = f(x)/S(x)$$

The derivation of these expressions and the theoretical analysis of the hazard rate function is provided in Appendix IV. Comparing Expressions (6.79) and (6.81) this combination is equivalent to two 'series stages in series with a distinctive stage' in parallel as shown in Fig. 6.12b. The various characteristics of this combination are illustrated in Fig. 6.13.

Determination of Parameters

After a model is chosen to approximate a distribution, the next problem is to find the model parameters to fit the distribution. There are generally no explicit formulae for directly deriving the approximate stage model parameters from those of the distribution. In many cases, the parameters that will best define an

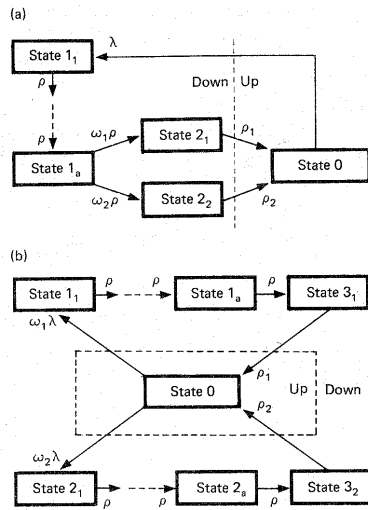


Fig. 6.12 The state transition diagram of a system with the down state represented by:
 (a) A series stage in series with two parallel stages
 (b) Two 'series stages in series with a distinctive stage' in parallel

empirical distribution are not known. The moments can, however, always be evaluated for any distribution either by exact or approximate methods. A method of determining the parameters for approximate stage models by matching the first r moments of the model and the distribution is presented in the following sections. This method is quite general in application.

The parameters of the stage model are non-linear and implicit functions of its moments. On the contrary, the first r moments for the stage combinations discussed in this paper can be easily calculated from the parameters. The Newton–Raphson method of successive approximation is applied to solve for the parameters from the given moments. This method requires for each stage of approximation:

1. evaluation of the moments with the given parameters
2. evaluation of the partial derivatives of the moments with respect to each parameter

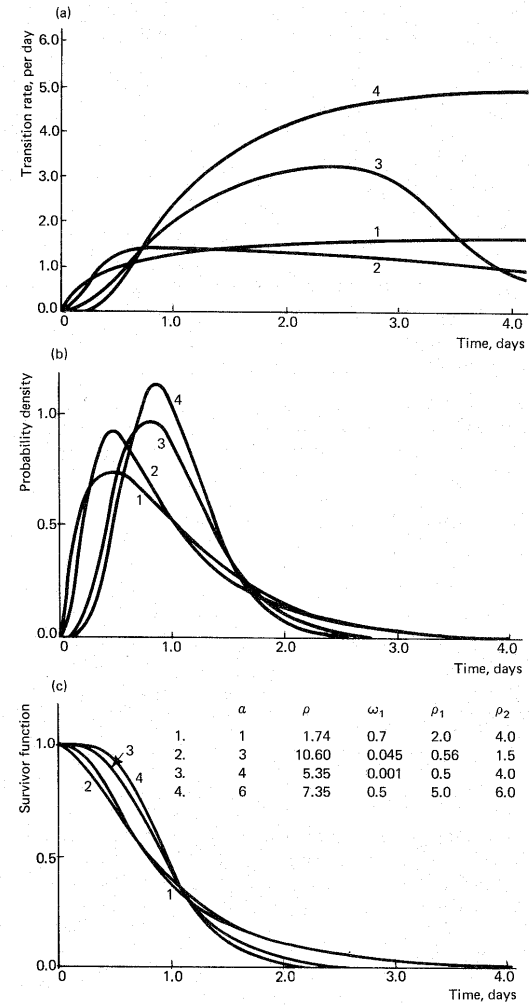


Fig. 6.13 Some characteristics of the distribution associated with series stages in series with two parallel stages.

1 Moment Evaluation for a Combination of Stages

The probability density functions of the stage models discussed have simple rational Laplace transforms. The r th moment about zero can be obtained if the r th derivative of the Laplace of the probability density function exists at $s = 0$. The r th moment m_r of the distribution is

$$m_r = (-1)^r \bar{f}^{(r)}(0)$$

where

$$\bar{f}^{(r)}(0) = \left. \frac{d^r \bar{f}(s)}{ds^r} \right|_{s=0}$$

$\bar{f}(s)$ being the Laplace transform of the probability density function. Moment calculations for some of the stage models are given in Appendix V.

2. Newton Raphson Method for Parameter Calculation

The first r moments are matched by successive approximation starting from the initial parameters. If a model has r parameters, x_1, x_2, \dots, x_r to be determined by matching the first r moments, the r functions, $\phi_1, \phi_2, \dots, \phi_r$ are defined such that

$$\phi_1 = \phi_1(X) = m_1(X) - M_1$$

$$\phi_2 = \phi_2(X) = m_2(X) - M_2$$

...

$$\phi_r = \phi_r(X) = m_r(X) - M_r$$

where X is the column vector $(x_1, x_2, \dots, x_r)^t$ and $m_r(X)$ is the r th moment of the stage model and M_r is the r th moment of the distribution to be approximated. The conditions of exact match of the first r moments are

$$\phi_1 = \phi_2 = \dots = \phi_r = 0$$

Let $X_0 = (x_{10}, x_{20}, \dots, x_{r0})^t$ be the vector of the parameters at a certain stage of approximation, and let ϕ be a column vector such that $\phi = (\phi_1, \phi_2, \dots, \phi_r)^t$. The correction vector for parameter $\Delta X = (\Delta x_1, \Delta x_2, \dots, \Delta x_r)^t$, can be calculated from the following matrix equation by the Gauss elimination method if $\phi(X_0)$ and $\phi'(X_0)$ are known.

$$\phi(X_0) + \phi'(X_0)\Delta X = 0$$

where $\phi(X_0)$ is the vector ϕ and $X = X_0$, $\phi'(X_0)$ is the Jacobian matrix of ϕ at X_0 , i.e.

$$\phi'(X_0) = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1}(X_0) & \frac{\partial \phi_1}{\partial x_2}(X_0) & \dots & \frac{\partial \phi_1}{\partial x_r}(X_0) \\ \frac{\partial \phi_2}{\partial x_1}(X_0) & \frac{\partial \phi_2}{\partial x_2}(X_0) & \dots & \frac{\partial \phi_2}{\partial x_r}(X_0) \\ \dots & \dots & \dots & \dots \\ \frac{\partial \phi_r}{\partial x_1}(X_0) & \frac{\partial \phi_r}{\partial x_2}(X_0) & \dots & \frac{\partial \phi_r}{\partial x_r}(X_0) \end{bmatrix}$$

The improved parameter values are obtained by $X = X_0 + \Delta X$. The $\phi(X_0)$ can be calculated directly from the first r moments of the model when $X = X_0$ (Appendix VI).

The method for finding $\phi'(X_0)$ is discussed for some of the stage models in Appendix VI.

Example system study

The technique of stages is applied to a two state unit having the up time exponentially distributed and down time with a lognormal distribution. The following numerical values are used

$$\text{Mean Up Time} = 1500 \text{ hr}$$

$$\text{Mean Down Time} = 20 \text{ hr}$$

The standard deviation of the down time is varied as

- (i) 10 hr
- (ii) 14.14 hr
- (iii) 20 hr

The lognormal distribution is completely specified by its mean and standard deviation. The expression for a lognormal probability density function of the random variable X was given in Chapter 2.

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-(\log x - m)^2 / 2\sigma^2}$$

where m and σ are the standard deviation of $\log(X)$. The r th moment of X is

$$m_r(X) = E(X^r) = e^{mr + \sigma^2 r^2 / 2} \tag{6.83}$$

The mean is

$$m_X = e^{m + \sigma^2 / 2} \tag{6.84}$$

and the variance is

$$\sigma_X^2 = e^{2m + 2\sigma^2} - e^{2m + \sigma^2} \tag{6.85}$$

Solving Equations (6.84) and (6.85)

$$\sigma^2 = \log \left[\left(\frac{\sigma_x}{m_x} \right)^2 + 1 \right]$$

and

$$m = \log m_x - \frac{1}{2} \log \left[\left(\frac{\sigma_x}{m_x} \right)^2 + 1 \right]$$

The parameters m and σ can therefore be found from the mean m and standard deviation σ_x of the log normal distribution. The hazard rate of a lognormal distribution shows initial positive ageing followed by negative ageing.

This suggests two combinations:

- (a) two series stages in parallel
- (b) series stages in series with two parallel stages

The log normal is approximated by these two combinations using moment matching technique and the parameters are listed in Table 6.1. The transition rate functions, probability density functions and the survivor functions of the log normal distributions together with the stage combination models are shown in Fig. 6.14

Table 6.1 The parameters of stage combinations for the approximation of lognormal distributions

The mean of the distribution $m_x = 20$ hours, $\sigma_x =$ Standard deviation of the distribution

(a) Approximation by series stages in series with two parallel stages

σ_x (hours)	Approximate model parameters				
	a	ω_1	ρ per hour	ρ_1 per hour	ρ_2 per hour
10	6	0.10976	0.526715	0.078325	0.123519
14.14	4	0.05270	0.554433	0.036597	0.083496
20	3	0.02250	1.221384	0.016180	0.060515

(b) Approximation by two series stages in parallel

σ_x (hours)	a_1	a_2	ω_1	ρ_1 per hour	ρ_2 per hour
	10	4	6	0.21601	0.144001
14.14	2	4	0.25464	0.066666	0.241201
20	2	2	0.06699	0.031698	0.118301

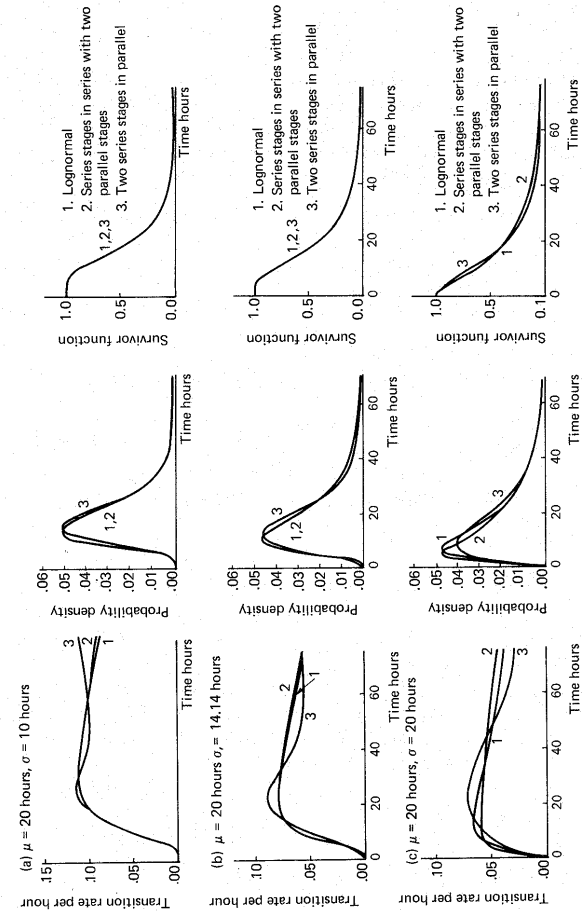


Fig. 6.14 Characteristics of lognormal distributions and their approximate stage combination models.

Several techniques exist for finding time specific probabilities of the states of Markovian process. The probability for the down state was evaluated up to 24 hours assuming the system was initially in the up state. The continuous time Markov process is approximated by a discrete time process using a small time interval and the state probabilities are obtained by multiplication of the transition probability matrix. The results are shown in Table 6.2. The results compare well with the direct approximate expression derived by assuming that the time period considered is so short that no more than one forced outage and repair can occur in it. Denoting the up and down states by 1 and 2 respectively, and assuming $p_1(0) = 1$

$$p_2(t) \approx p_2^a(t) = P(\text{one forced outage and no repair in } t) \\ = \int_0^t \lambda e^{-\lambda x} S_d(t-x) dx$$

where

λ = failure rate

S_d = survivor function of down time

and

$p_2^a(t)$ = the approximate expression for the probability of being in the down state at time t

Assuming the interval $(0, t)$ to be divided into k equal subintervals each of length δ

$$p_2^a(t) = \sum_{i=1}^k [P(\text{forced outage in } i\text{th interval})P(\text{no repair in interval of length } (k-i)\delta)] \\ = \sum_{i=1}^k [e^{-\lambda(i-1)\delta} - e^{-\lambda i\delta}] S_d\{(k-i)\delta\}$$

when $\lambda t \ll 1$

$$e^{-\lambda t} = 1 - \lambda t$$

Therefore

$$p_2^a(t) = \lambda \delta \sum_{i=1}^k S_d\{(k-i)\delta\} \\ = \lambda \delta \sum_{i=1}^k S_d\{(i-1)\delta\} \\ = \lambda \delta \sum_{i=0}^{k-1} S_d\{i\delta\}$$

It may be necessary to find the state probability values at the end of each equal time interval, i.e. in a 24 hr period, calculate $p_2^a(t)$ at the end of each

hour. The probability at the end of the n th interval is given by

$$p_2^a(nm\delta) = p_2^a\{(n-1)m\delta\} + \lambda \delta \sum_{i=(n-1)m}^{nm-1} S_d(i\delta)$$

where m is the number of subintervals in each time interval and δ is the length of each subinterval, the length of each interval therefore being $m\delta$. The closed form expression for the survivor function of the lognormal distribution does not exist and Simpson's Rule can be applied to calculate this by integrating the probability density function. The following equation can be used to calculate $S_d(i\delta)$ successively

$$S_d(i\delta) = S\{(i-1)\delta\} - \frac{\delta}{6l} \sum_{j=0}^{l-1} \left[f\left\{(i-1)\delta + \frac{j\delta}{l}\right\} \right. \\ \left. + 4f\left\{(i-1)\delta + \left(j + \frac{1}{2}\right)\frac{\delta}{l}\right\} + f\left\{(i-1)\delta + (j+1)\frac{\delta}{l}\right\} \right]$$

where l is the number of divisions in each subinterval for calculation of the survivor function by Simpson's Rule. This method was developed for computer application without requiring excessive storage. Down state probabilities are evaluated up to 24 hours with a one hour interval. They are listed in Table 6.2 together with those obtained by the device of stages. It can be seen that the results of the two approximate methods are quite close to each other.

Application to The Transformer Bank Problem

Reliability Modelling for Three Single Phase Transformers With One Spare, General distributions for Repair and Change Out, Restricted Repair, Using Stages in Parallel.

The expressions for this model assuming general distributions have already been developed. However, to illustrate the modelling approach when the stages are in parallel, the state transition diagram is developed in Fig. 6.15. Since the change out operation is restricted to a single stage, from the point of view of steady state analysis, the change out rate can be simply represented by an average value γ .

The probability density function for the repair time is

$$\omega_1 \mu_1 e^{-\mu_1 x} + \omega_2 \mu_2 e^{-\mu_2 x} \quad \text{with mean} = \omega_1/\mu_1 + \omega_2/\mu_2 \quad (6.86)$$

The state transition diagram in Fig. 6.15 can be clarified by referring to the discussion related to Fig. 6.7. The first letter of the state numbers represents

Table 6.2. Time specific probabilities of the down state calculated by using

(a) The approximate expression (b) The approximate 'series stages in series with two parallel stages' (c) the approximate 'two series stages in parallel'. The Mean Down Time = 20 hours. The Mean Up Time = 1500 hours

Time (hours)	Std. deviation = 10 hours			Std. deviation = 14.4 hours			Std. deviation = 20 hours		
	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
1	6.667 × 10 ⁻⁴	6.667 × 10 ⁻⁴	6.667 × 10 ⁻⁴	6.667 × 10 ⁻⁴	6.674 × 10 ⁻⁴	6.670 × 10 ⁻⁴	6.666 × 10 ⁻⁴	6.680 × 10 ⁻⁴	6.654 × 10 ⁻⁴
2	1.333 × 10 ⁻³	1.334 × 10 ⁻³	1.334 × 10 ⁻³	1.333 × 10 ⁻³	1.334 × 10 ⁻³	1.332 × 10 ⁻³	1.331 × 10 ⁻³	1.330 × 10 ⁻³	1.323 × 10 ⁻³
3	2.000 × 10 ⁻³	2.000 × 10 ⁻³	2.000 × 10 ⁻³	2.000 × 10 ⁻³	1.998 × 10 ⁻³	1.998 × 10 ⁻³	1.985 × 10 ⁻³	1.974 × 10 ⁻³	1.966 × 10 ⁻³
4	2.008 × 10 ⁻³	2.686 × 10 ⁻³	2.666 × 10 ⁻³	2.000 × 10 ⁻³	2.658 × 10 ⁻³	2.651 × 10 ⁻³	2.620 × 10 ⁻³	2.591 × 10 ⁻³	2.590 × 10 ⁻³
5	3.800 × 10 ⁻³	3.338 × 10 ⁻³	3.328 × 10 ⁻³	3.312 × 10 ⁻³	3.308 × 10 ⁻³	3.297 × 10 ⁻³	3.280 × 10 ⁻³	3.178 × 10 ⁻³	3.192 × 10 ⁻³
6	3.000 × 10 ⁻³	3.988 × 10 ⁻³	3.986 × 10 ⁻³	3.950 × 10 ⁻³	3.943 × 10 ⁻³	3.930 × 10 ⁻³	3.811 × 10 ⁻³	3.732 × 10 ⁻³	3.770 × 10 ⁻³
7	4.650 × 10 ⁻³	4.640 × 10 ⁻³	4.636 × 10 ⁻³	4.567 × 10 ⁻³	4.557 × 10 ⁻³	4.545 × 10 ⁻³	4.362 × 10 ⁻³	4.255 × 10 ⁻³	4.321 × 10 ⁻³
8	5.800 × 10 ⁻³	5.280 × 10 ⁻³	5.278 × 10 ⁻³	5.160 × 10 ⁻³	5.147 × 10 ⁻³	5.139 × 10 ⁻³	4.880 × 10 ⁻³	4.748 × 10 ⁻³	4.846 × 10 ⁻³
9	5.928 × 10 ⁻³	5.983 × 10 ⁻³	5.895 × 10 ⁻³	5.725 × 10 ⁻³	5.709 × 10 ⁻³	5.708 × 10 ⁻³	5.367 × 10 ⁻³	5.212 × 10 ⁻³	5.342 × 10 ⁻³
10	6.560 × 10 ⁻³	6.595 × 10 ⁻³	6.496 × 10 ⁻³	6.268 × 10 ⁻³	6.240 × 10 ⁻³	6.249 × 10 ⁻³	5.823 × 10 ⁻³	5.650 × 10 ⁻³	5.812 × 10 ⁻³
11	7.110 × 10 ⁻³	7.081 × 10 ⁻³	7.072 × 10 ⁻³	6.765 × 10 ⁻³	6.740 × 10 ⁻³	6.761 × 10 ⁻³	6.250 × 10 ⁻³	6.061 × 10 ⁻³	6.254 × 10 ⁻³
12	7.000 × 10 ⁻³	7.628 × 10 ⁻³	7.620 × 10 ⁻³	7.288 × 10 ⁻³	7.208 × 10 ⁻³	7.243 × 10 ⁻³	6.649 × 10 ⁻³	6.449 × 10 ⁻³	6.669 × 10 ⁻³
13	8.178 × 10 ⁻³	8.143 × 10 ⁻³	8.137 × 10 ⁻³	7.680 × 10 ⁻³	7.644 × 10 ⁻³	7.694 × 10 ⁻³	7.022 × 10 ⁻³	6.815 × 10 ⁻³	7.059 × 10 ⁻³
14	8.681 × 10 ⁻³	8.626 × 10 ⁻³	8.021 × 10 ⁻³	8.092 × 10 ⁻³	8.050 × 10 ⁻³	8.114 × 10 ⁻³	7.370 × 10 ⁻³	7.159 × 10 ⁻³	7.424 × 10 ⁻³
15	9.109 × 10 ⁻³	9.068 × 10 ⁻³	9.070 × 10 ⁻³	8.475 × 10 ⁻³	8.427 × 10 ⁻³	8.503 × 10 ⁻³	7.696 × 10 ⁻³	7.483 × 10 ⁻³	7.765 × 10 ⁻³
16	9.522 × 10 ⁻³	9.477 × 10 ⁻³	9.488 × 10 ⁻³	8.830 × 10 ⁻³	8.776 × 10 ⁻³	8.862 × 10 ⁻³	8.000 × 10 ⁻³	7.789 × 10 ⁻³	8.088 × 10 ⁻³
17	9.901 × 10 ⁻³	9.852 × 10 ⁻³	9.863 × 10 ⁻³	9.160 × 10 ⁻³	9.099 × 10 ⁻³	9.193 × 10 ⁻³	8.285 × 10 ⁻³	8.077 × 10 ⁻³	8.380 × 10 ⁻³
18	1.025 × 10 ⁻²	1.019 × 10 ⁻²	1.021 × 10 ⁻²	9.464 × 10 ⁻³	9.398 × 10 ⁻³	9.498 × 10 ⁻³	8.551 × 10 ⁻³	8.318 × 10 ⁻³	8.657 × 10 ⁻³
19	1.056 × 10 ⁻²	1.050 × 10 ⁻²	1.052 × 10 ⁻²	9.746 × 10 ⁻³	9.674 × 10 ⁻³	9.776 × 10 ⁻³	8.800 × 10 ⁻³	8.604 × 10 ⁻³	8.916 × 10 ⁻³
20	1.085 × 10 ⁻²	1.078 × 10 ⁻²	1.080 × 10 ⁻²	1.001 × 10 ⁻²	9.993 × 10 ⁻³	1.008 × 10 ⁻²	9.084 × 10 ⁻³	8.845 × 10 ⁻³	9.155 × 10 ⁻³
21	1.111 × 10 ⁻²	1.104 × 10 ⁻²	1.106 × 10 ⁻²	1.025 × 10 ⁻²	1.017 × 10 ⁻²	1.026 × 10 ⁻²	9.253 × 10 ⁻³	9.072 × 10 ⁻³	9.878 × 10 ⁻³
22	1.134 × 10 ⁻²	1.126 × 10 ⁻²	1.129 × 10 ⁻²	1.047 × 10 ⁻²	1.038 × 10 ⁻²	1.048 × 10 ⁻²	9.458 × 10 ⁻³	9.286 × 10 ⁻³	9.586 × 10 ⁻³
23	1.155 × 10 ⁻²	1.146 × 10 ⁻²	1.149 × 10 ⁻²	1.067 × 10 ⁻²	1.058 × 10 ⁻²	1.067 × 10 ⁻²	9.650 × 10 ⁻³	9.486 × 10 ⁻³	9.778 × 10 ⁻³
24	1.174 × 10 ⁻²	1.165 × 10 ⁻²	1.167 × 10 ⁻²	1.085 × 10 ⁻²	1.077 × 10 ⁻²	1.085 × 10 ⁻²	9.881 × 10 ⁻³	9.678 × 10 ⁻³	9.958 × 10 ⁻³

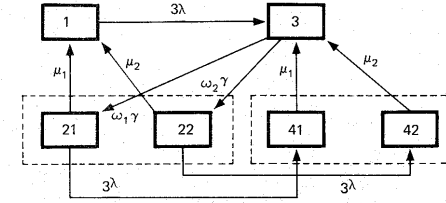


Fig. 6.15 The state transition diagram for three single phase transformers with one spare, assuming restricted repair and two stages in parallel for the repair time.

the system state as given in Fig. 6.2, and the second letter if any denotes the stage number. The steady state equations for this model can be written in the conventional manner.

The solution of these equations gives

$$p_1 = 1/Z \quad \text{where} \quad Z = 1 + \frac{3\lambda}{E} \left(\frac{1}{\gamma} + \frac{1}{\mu} \right)$$

where

$$E = \frac{\omega_1 \mu_1}{3\lambda + \mu_1} + \frac{\omega_2 \mu_2}{3\lambda + \mu_2} \quad \text{and} \quad \frac{1}{\mu} = \frac{\omega_1}{\mu_1} + \frac{\omega_2}{\mu_2}$$

also

$$p_2 = p_{21} + p_{22} = \frac{1-E}{ZE}, \quad p_3 = \frac{3\lambda}{\gamma EZ} \quad \text{and} \quad p_4 = p_{41} + p_{42} = \frac{9\gamma^2}{ZE} \left(\frac{\omega_1}{\mu_1(3\lambda + \mu_1)} + \frac{\omega_2}{\mu_2(3\lambda + \mu_2)} \right)$$

Three Single Phase Transformers With a Single Spare, Unrestricted Repair, Erlangian Distribution for Repair and Reinstallation.

The state transition diagram shown in Fig. 6.16, assumes three stages for each of the repair and reinstallation periods, i.e. the index 'a' in Equation (6.68) is 3. The transition diagrams for higher 'a' can be similarly developed and the numerical answers when a is a non-integer, (i.e. when Equation (6.69) is assumed as the distribution) can be obtained by interpolation. A computer program can be written which can generate the transition rate matrix for any 'a' without having to draw the state transition diagram. The reliability evaluation can, therefore, be made conveniently for any value of 'a'.

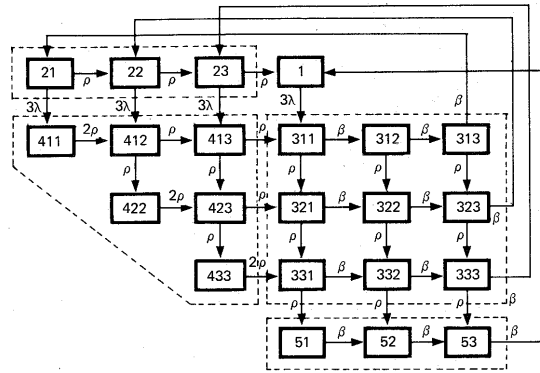


Fig. 6.16 The state transition diagram for three single phase transformers with a single spare assuming unrestricted repair and a general Erlangian distribution for the repair and reinstallation period.

The repair and the change out rates associated with each stage are assumed as ρ and β respectively such that $\rho = a.\mu$ and $\beta = a.\gamma$. The first letter of the state number in Fig. 6.16 indicates the system state as shown in Fig. 6.1 and the remaining letters denote the stages of the operations being carried out. For example, 22 means that the system is in state 2 (the transformer bank is up and no spare) and the repair is in the second stage. Similarly, 412 indicates state 4 (transformer bank failed and no spare) and the repair on one of the transformers is in the first stage and the other transformer in the second stage. As another example, 323 denotes state 3 (transformer bank down and one spare) and the repair is in the second stage whereas the change out operation is in stage 3. The relationship between the state and stage availabilities is

$$p_1 = p_1, \quad p_2 = \sum_{i=1}^a p_{2i}, \quad p_3 = \sum_{i=1}^a \sum_{j=1}^a p_{3ij},$$

$$p_4 = \sum_{i=1}^a \sum_{j=1}^a p_{4ij}, \quad p_5 = \sum_{i=1}^a p_{5i}$$

The steady state equations for Fig. 6.16 can be written in the form

$$AP = B \tag{6.87}$$

where

$B = A$ column vector with all zeros

$P =$ the vector of steady state probabilities

and

$A =$ the transpose of the transition rate matrix

Any $(a - 1)$ equations out of a in (6.87) can be solved with the normalizing equation

$$\sum_{i=1}^5 p_i = 1$$

to give the steady state probabilities.

Numerical Results

These studies were conducted to determine the effect of the distribution form on the steady state probabilities. The exponential distribution was assumed for the up time and the repair and change out periods were assumed to have the Special Erlangian distribution of Equation (6.68) with the same shape parameter 'a'.

Restricted Repair

The failure rate was taken as 0.008 failures per year and the effect of variation of 'a' on the probability of being in the down state was determined under different values for the mean repair and change out times using Equation (6.30). The results are shown in Tables 6.3 and 6.4. It should be noted that the mean values are held constant so that the difference in values is entirely due to the change in the shape of the distribution. The values with $a = 1$ correspond to the exponential distribution and the limiting values (LV) refer to the constant repair and reinstallation times i.e. when $a \rightarrow \infty$ with the mean values kept constant.

Table 6.3 shows the actual values of the steady state unavailability and Table 6.4 shows the percentage variation from the exponential when the value of 'a' is increased. It can be seen that the value of 'a' has a considerable effect on the unavailability of the transformer bank. The variation from the exponential depends both upon the mean repair time (MRT) and the mean change out time (MIT). For a given mean change out time, the greater the mean repair time the more pronounced is the variation from the exponential and for a given mean repair time, the greater the mean change out time the less pronounced the variation.

Table 6.3 The effect of *a* on the unavailability of the transformer bank

a	M.R.T. = 182.5 Days		M.R.T. = 20 Days	
	M.I.T. = 0.5 (Days)	M.I.T. = 3.5 (Days)	M.I.T. = 0.5 (Days)	M.I.T. = 3.5 (Days)
1	0.175139×10^{-3}	0.372291×10^{-3}	0.346026×10^{-4}	0.231810×10^{-3}
2	0.140000×10^{-3}	0.337166×10^{-3}	0.341714×10^{-4}	0.231379×10^{-3}
3	0.128224×10^{-3}	0.325395×10^{-3}	0.340276×10^{-4}	0.231236×10^{-3}
4	0.122325×10^{-3}	0.319498×10^{-3}	0.339557×10^{-4}	0.231164×10^{-3}
5	0.118781×10^{-3}	0.315956×10^{-3}	0.339126×10^{-4}	0.231121×10^{-3}
LV	0.104546×10^{-3}	0.301726×10^{-3}	0.337212×10^{-4}	0.230929×10^{-3}

Table 6.4 The percent variation from the values of exponential

a	M.R.T. = 182.5 Days		M.R.T. = 20 Days	
	M.I.T. = 0.5 (Days)	M.I.T. = 3.5 (Days)	M.I.T. = 0.5 (Days)	M.I.T. = 3.5 (Days)
1	0.00	0.00	0.00	0.00
2	20.06	9.43	1.25	0.18
3	26.79	12.60	1.66	0.25
4	30.16	14.18	1.87	0.28
5	32.18	15.13	1.99	0.30
LV	40.31	18.95	2.55	0.38

Unrestricted Repair

A similar study was conducted for unrestricted repair using the computer program which generates the transition rate matrix for different values of 'a' and evaluates the various steady state probabilities. The variation of the unavailability with the increase in 'a' is shown in Table 6.5A and similar variations in the states constituting the failure state, i.e. p_3, p_4 and p_5 are shown in Tables 6.5B – D. The probability density functions for both repair and change out are assumed to be Special Erlangian.

The exponential distribution, as in the case of restricted repair, does overestimate the unavailability but the variation is insignificant. The individual components of the failure state show an interesting behaviour; p_3 shows an increase with increasing 'a' whereas p_4 decreases. The only component which shows a large variation is p_5 , however, its magnitude is relatively quite small. The large variation in the value of p_5 is explained by the fact that an increase in 'a' is accompanied by a rapid decrease of dispersion of the repair and change

out times which results in a lower probability of being repaired while the change out is in progress. The overall variation in the unavailability is, however, quite small.

Two Three Phase Transformers in Parallel, Special Erlangian Distribution for Repair and Change Out Periods

The combination of stages to approximate the probability density function depends upon the available information. However, a model has been developed

Table 6.5 The variation in the probability of the failure state and its constituent states

a	M.R.T. = 182.5 Days		M.R.T. = 20 Days	
	M.I.T. = 0.5 (Days)	M.I.T. = 3.5 (Days)	M.I.T. = 0.5 (Days)	M.I.T. = 3.5 (Days)
A. Variation in Unavailability				
1	0.103818×10^{-3}	0.299854×10^{-3}	0.337181×10^{-4}	0.230819×10^{-3}
2	0.103769×10^{-3}	0.299513×10^{-3}	0.337126×10^{-4}	0.230780×10^{-3}
3	0.103745×10^{-3}	0.299344×10^{-3}	0.337099×10^{-4}	0.230760×10^{-3}
4	0.103731×10^{-3}	0.299237×10^{-3}	0.337081×10^{-4}	0.230748×10^{-3}
5	0.103720×10^{-3}	0.299164×10^{-3}	0.337068×10^{-4}	0.230739×10^{-3}
B. Variation in p_3				
1	0.327835×10^{-4}	0.225739×10^{-3}	0.320738×10^{-4}	0.195816×10^{-3}
2	0.328720×10^{-4}	0.229870×10^{-3}	0.328363×10^{-4}	0.220615×10^{-3}
3	0.328726×10^{-4}	0.230027×10^{-3}	0.328726×10^{-4}	0.226905×10^{-3}
4	0.328726×10^{-4}	0.230037×10^{-3}	0.328748×10^{-4}	0.228900×10^{-3}
5	0.328726×10^{-4}	0.230039×10^{-3}	0.328749×10^{-4}	0.229609×10^{-3}
C. Variation in p_4				
1	0.709445×10^{-4}	0.697866×10^{-4}	0.842477×10^{-6}	0.734782×10^{-6}
2	0.708962×10^{-4}	0.694447×10^{-4}	0.836972×10^{-6}	0.695249×10^{-6}
3	0.708722×10^{-4}	0.692747×10^{-4}	0.834234×10^{-6}	0.674866×10^{-6}
4	0.708574×10^{-4}	0.691686×10^{-4}	0.832526×10^{-6}	0.662111×10^{-6}
5	0.708467×10^{-4}	0.690945×10^{-4}	0.831331×10^{-6}	0.653233×10^{-6}
D. Variation in p_5				
1	0.898177×10^{-7}	0.432924×10^{-5}	0.801844×10^{-6}	0.342618×10^{-4}
2	0.128310×10^{-8}	0.197985×10^{-6}	0.393765×10^{-7}	0.846884×10^{-5}
3	0.713079×10^{-9}	0.415356×10^{-7}	0.301219×10^{-8}	0.318000×10^{-5}
4	0.665219×10^{-9}	0.323045×10^{-7}	0.808461×10^{-9}	0.118599×10^{-5}
5	0.638378×10^{-9}	0.305947×10^{-7}	0.636642×10^{-9}	0.477498×10^{-5}

in Fig. 6.17, assuming the Special Erlangian distribution for repair and change out because by suitably varying the parameter 'a' the behaviour of a large number of probability density functions less dispersed than the exponential can be approximated. The same symbols and notation as in Fig. 6.16 has been used. The transition diagram shown is for three stages. A computer program can be

easily written to generate the transition rate matrix for any value of 'a'. The numerical values for time dependent or steady state probabilities can then be obtained.

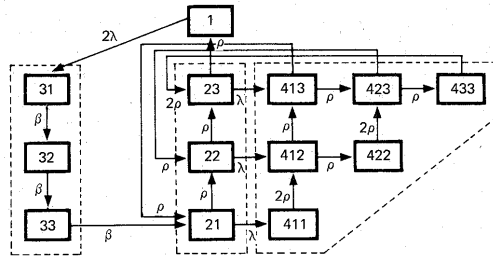


Fig. 6.17 The state transition diagram for two three phase half capacity transformers in parallel assuming the special Erlangian distribution for repair and change out times.

The Special Case of Independent Components

If a system is composed of independent binary components, the associated stochastic process is a superposition of independent alternating renewal processes. It was proved in Chapter 3 for an equilibrium alternating renewal process that irrespective of the probability density function of the up and down time durations, the frequency of encountering the up and down states is given by

$$f_u = f_d = \frac{1}{T_u + T_d}$$

It is also known that for an equilibrium alternating renewal process, the probabilities of being in the up and down state respectively are given by

$$p_u = \frac{T_u}{T_u + T_d}$$

and

$$p_d = \frac{T_d}{T_u + T_d}$$

The transition rates from up to down, λ , and down to up, μ , are therefore, given by

$$\lambda = f_d/p_u = \frac{1}{T_u} \quad \text{and} \quad \mu = f_u/p_d = \frac{1}{T_d}$$

It is therefore clear that under steady state conditions, the interstate transition rates of an alternating renewal process can be represented by the reciprocals of the respective mean state durations. Since the different alternating renewal processes are independent, the steady state probabilities and frequencies will be unaffected by the forms of the probability density functions of the state durations provided the transition rates are represented by the mean component state durations.

Reliability Modelling Using Complex Transition Rates

It has been noted that when complex transition rates are allowed, any distribution having a rational Laplace transform can, in principle, be treated using the state device. The use of complex transition rates is illustrated for a component whose up time is exponentially distributed with rate parameter λ and down time has the density function

$$f(x) = a \frac{(a^2 + b^2)}{b^2} e^{-at} (1 - \cos bt) \tag{6.88}$$

The Laplace transform is

$$\begin{aligned} \bar{f}(s) &= \frac{a}{a+s} \frac{a^2 + b^2}{(a+s)^2 + b^2} \\ &= \frac{a}{a+s} \frac{a+ib}{(a+ib)+s} \frac{a-ib}{(a-ib)+s} \end{aligned}$$

This expression is the product of the Laplace of three functions as shown below

Laplace function	function
$\frac{a}{a+s}$	$a e^{-at}$
$\frac{a+ib}{(a+ib)+s}$	$(a+ib) e^{-(a+ib)t}$
$\frac{a-ib}{(a-ib)+s}$	$(a-ib) e^{-(a-ib)t}$

Equation (6.88), therefore, is the probability density function of the random variable which is the sum of three random variables having exponential distributions. The state transition diagram of this component is, therefore, as

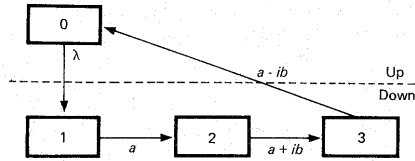


Fig. 6.18 State transition diagram of a component having an exponential up time distribution and a down time distribution given by Equation (6.88)

shown in Fig. 6.18. The state differential equations can be written as below

$$p'_0(t) = -\lambda p_0(t) + (a - ib)p_3(t)$$

$$p'_1(t) = -ap_1(t) + \lambda p_0(t)$$

$$p'_2(t) = -(a + ib)p_2(t) + ap_1(t)$$

$$p'_3(t) = -(a - ib)p_3(t) + (a + ib)p_2(t)$$

Assuming $p_0(0) = 1$, the Laplace transforms of the above equations are

$$sp_0(s) - 1 = -\lambda p_0(s) + (a - ib)p_3(s) \quad (6.89)$$

$$sp_1(s) = -ap_1(s) + \lambda p_0(s) \quad (6.90)$$

$$sp_2(s) = -(a + ib)p_2(s) + ap_1(s) \quad (6.91)$$

$$sp_3(s) = -(a - ib)p_3(s) + (a + ib)p_2(s) \quad (6.92)$$

From Equations (6.90)–(6.92)

$$p_1(s) = \frac{\lambda}{s + a} p_0(s) \quad (6.93)$$

$$\begin{aligned} p_2(s) &= \frac{a}{s + a + ib} p_1(s) \\ &= \frac{a\lambda}{(s + a)(s + a + ib)} p_0(s) \end{aligned} \quad (6.94)$$

$$p_3(s) = \frac{a + ib}{s + (a - ib)} \frac{a\lambda}{(s + a)(s + a + ib)} p_0(s) \quad (6.95)$$

Substituting (6.95) into (6.89)

$$\begin{aligned} (s + \lambda)p_0(s) &= 1 + (a - ib)p_3(s) \\ &= 1 + \frac{(a - ib)(a + ib)a\lambda}{(s + a - ib)(s + a + ib)(s + a)} p_0(s) \end{aligned}$$

$$\left[(s + \lambda) - \frac{(a^2 + b^2)a\lambda}{(s + a)((s + a)^2 + b^2)} \right] p_0(s) = 1$$

Therefore

$$p_0(s) = \frac{(s + a)((s + a)^2 + b^2)}{s\{(s + a)^2 + b^2\}[s + a + \lambda] + a\lambda(s + 2a)}$$

$$p_1(s) = \frac{\lambda((s + a)^2 + b^2)}{s\{(s + a)^2 + b^2\}[s + a + \lambda] + a\lambda(s + 2a)}$$

$$p_2(s) = \frac{a\lambda(s + a - ib)}{s\{(s + a)^2 + b^2\}[s + a + \lambda] + a\lambda(s + 2a)}$$

$$p_3(s) = \frac{a\lambda(a + ib)}{s\{(s + a)^2 + b^2\}[s + a + \lambda] + a\lambda(s + 2a)}$$

$$\begin{aligned} p_{DN}(s) &= p_1(s) + p_2(s) + p_3(s) \\ &= \frac{\lambda((s + a)^2 + b^2) + a\lambda(s + a - ib) + a\lambda(a + ib)}{s\{\cdot\}} \end{aligned}$$

$$= \frac{\lambda((s + a)^2 + b^2) + a\lambda(s + 2a)}{s\{(s + a)^2 + b^2\}[s + a + \lambda] + a\lambda(s + 2a)}$$

It can be seen that $p_2(s)$ and $p_3(s)$ when inverted will yield complex probabilities but $p_{DN}(t)$ will be real.

Steady State

$$\begin{aligned} p_{DN}(t) &= \lim_{t \rightarrow \infty} \lim_{s \rightarrow 0} sp_{DN}(s) \\ &= \frac{\lambda(a^2 + b^2) + 2a^2\lambda}{(a^2 + b^2)(a + \lambda) + 2a^2\lambda} \\ &= \frac{\lambda[a^2 + b^2 + 2a^2]}{\lambda(a^2 + b^2 + 2a^2) + a(a^2 + b^2)} \\ &= \frac{\lambda}{\lambda + \frac{a(a^2 + b^2)}{b^2 + 3a^2}} \end{aligned}$$

$$= \frac{\lambda}{\lambda + \mu}$$

$$\text{where } \mu = \frac{a(a^2 + b^2)}{b^2 + 3a^2}$$

It can be proved that $\mu = \frac{1}{T_d}$

where T_d = The mean down time.

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