CHAPTER 3

Frequency and Associated Concepts

Introduction

This chapter develops the frequency balancing approach to the stochastic process. This method has been used with considerable success in the field of power system reliability evaluation and is known as the frequency and duration approach. The concept of frequency is examined in detail and used to derive expressions for the mean cycle time and the mean duration of a state. In Chapter 5 the concept of frequency is used to derive conditions of mergeability which are very useful in dealing with large systems.

Interstate Transition Rate

Consider a large number of identical systems having the underlying stochastic process $Z(t)$. All the systems start in state i and further assume that in one step, state i can communicate with either state j or state k. If the histories of all the systems are now plotted they may appear as shown in Fig. 3.1. These observations are called independent realizations of the stochastic process $Z(t)$. Further, if N realizations in which the system transits to state j are separated out, the durations of the system in state i in this subset are called N independent realizations of the random variable X_{ij}, i.e. the duration of the state i under the condition that the system will transit to state j. In the stochastic process where all the states intercommunicate, these realizations could be obtained from a single long realization of the system by observing from the moment of each entry into state i until the termination of state i by transiting to state j.

Define:

$$f_{ij}(x) = \text{The probability density function of the random variable } X_{ij}$$

$$F_{ij}(x) = \text{The distribution function of the random variable } X_{ij}$$

$$= P(X_{ij} \leqslant x)$$

$$= \int_0^x f_{ij}(y)\,dy$$

and

$$S_{ij}(x) = \text{The survivor function of the random variable } X_{ij}$$
$$= 1 - F_{ij}(x)$$
$$= \int_x^\infty f_{ij}(y)\,dy$$

Now introduce the random variable

$$n(x) = \text{The number of realizations in which the system is in state } i \text{ at time } x.$$

Then

$$E\{n(x)\} = \text{The expected value of } n(x)$$
$$= N \int_x^\infty f_{ij}(y)\,dy \qquad (3.1)$$
$$= \eta(x)$$

The expected number of transitions to state j in the interval $(x, x + \Delta x)$ is given by

$$\Delta\eta(x) = \eta(x) - \eta(x + \Delta x)$$

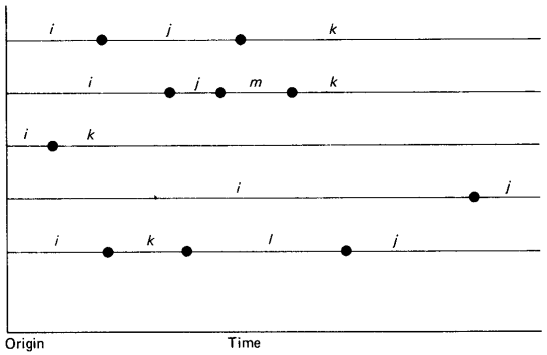

Fig. 3.1. Independent realizations of the stochastic process $Z(t)$.

and the rate of transition from state i to state j, per realization surviving up to x is

$$\lambda_{ij}(x) = \frac{\Delta\eta(x)}{\eta(x)} \cdot \frac{1}{\Delta x}$$
$$\Delta x \to 0^+$$
$$= -\frac{1}{\eta(x)} \frac{d\eta(x)}{dx} \qquad (3.2)$$

Substituting the values of $\eta(x)$ and $\frac{d\eta(x)}{dx}$ from (3.1)

$$\lambda_{ij}(x) = \frac{f_{ij}(x)}{S_{ij}(x)} \qquad (3.3)$$

The transition rate from state i to j is therefore the same as the hazard rate associated with the random variable X_{ij}. The above derivation is useful in getting an appreciation of the time specific transition rate as a relative expected rate. It is, however interesting to transform Equation (3.3) into another form.

$$\lambda_{ij}(x) = \frac{f_{ij}(x)}{S_{ij}(x)}$$
$$= \lim_{\Delta x \to 0^+} \frac{f_{ij}(x)\Delta x}{\Delta x} \frac{1}{S_{ij}(x)}$$
$$= \lim_{\Delta x \to 0^+} \frac{P(x < X_{ij} \leqslant x + \Delta x)}{\Delta x} \frac{1}{P(x < X_{ij})}$$
$$= \lim_{\Delta x \to 0^+} \frac{P(x < X_{ij} \leqslant x + \Delta x \,|\, x < X_{ij})}{\Delta x} \frac{P(x < X_{ij})}{P(x < X_{ij})}$$
$$= \lim_{\Delta x \to 0^+} \frac{P(x < X_{ij} \leqslant x + \Delta x \,|\, x < X_{ij})}{\Delta x} \qquad (3.4)$$

That is, as $\Delta x \to 0^+$

$$\lambda_{ij}(x)\Delta x = P(x < X_{ij} \leqslant x + \Delta x \,|\, x < X_{ij})$$
$$= \text{The probability of transiting from state } i \text{ to state } j \text{ in the interval } (x + \Delta x) \text{ given that this transition has not taken place up to time } x$$
$$= \text{The probability of a single transition from state } i \text{ to state } j \text{ at the age } x \text{ of state } i$$

Therefore, as $\Delta x \to 0^+$

The expected number of transitions from state i to state j in the interval $(x, x + \Delta x)$

$$= 1 . P(x < X_{ij} \leqslant x + \Delta x \mid x < X_{ij})$$

$$= \lambda_{ij}(x)\Delta x$$

i.e. $\lambda_{ij}(x)$ = The expected or the mean transition rate from state i to state j at the age x of state i.

The quantity $\lambda_{ij}(x)$ is called the age specific transition rate and under particular conditions may be designated as the failure rate, hazard function, etc. The concept of the interstate transition rate will be further treated in the next chapter while dealing with the state transition diagram. When X_{ij} is exponentially distributed with probability density function $\rho \exp\{-\rho x\}$, the age specific transition rate, as shown in the last chapter, is constant, i.e.

$$\lambda_{ij}(x) = \lambda_{ij} = \rho = \frac{1}{\text{Mean value of } X_{ij}}$$

The transition rate is, therefore, constant, i.e. independent of the age of the state and is equal to the reciprocal of the mean of the random variable X_{ij}. This property is true only for the exponential distribution when all the random variables generating the stochastic process are exponentially distributed, the transition rates are constant and the process is Markovian. The case of constant transition rates will be treated in detail in this chapter. The discussion of non-Markovian processes is deferred to Chapter 6.

The Concept of Frequency

The state space X of the stochastic process $Z(t)$ is assumed to be partitioned into two disjoint subsets X^+ and X^-. If any state of the subset is entered, that subset is said to have been encountered.
Define:

$f_+(t)$ = The time specific frequency of encountering the subset X^+. This is the expected rate at which X^+ is encountered at time t. For $\Delta t \to 0^+$, $f_+(t)\Delta t$ represents the expected number of times X^+ is encountered in the interval $(t, t+\Delta t)$.

$E_{ij}(t)$ = The time specific frequency of encountering state j from state i. This is the expected rate at which state j is encountered from state i or the mean rate at which the system transfers from state i to state j at time t.

$p_i(t)$ = The probability of being in state i at time t, for the given initial condition.

$p_+(t)$ = The probability of the system being in X^+ at time t for the given initial conditions.

$$= \sum_{i \in X^+} p_i(t)$$

λ_{ij} = The transition rate from state i to state j.

State j is encountered from state i at a rate λ_{ij} if the system is in state i, but if the system is not in state i then this rate is obviously zero. The transition rate or the encounter rate can therefore be represented by a discrete random variable $\beta_{ij}(t)$ such that

$$\beta_{ij}(t) = \begin{cases} \lambda_{ij} \text{ if } Z(t) = i \\ 0 \text{ if } Z(t) \neq i \end{cases}$$

$$E_{ij}(t) = \lambda_{ij} P\{Z(t) = i\} + 0 . P\{Z(t) \neq i\}$$

$$= \lambda_{ij} p_i(t)$$

Since the states are all mutually exclusive

$$f_+(t) = \sum_{i \in X^-} \sum_{j \in X^+} E_{ij}(t)$$

$$= \sum_{i \in X^-} \sum_{j \in X^+} p_i(t)\lambda_{ij} \tag{3.5}$$

If there is only one state j in X^+, then the time specific frequency of encountering it is

$$f_j(t) = \sum_{i \neq j} p_i(t)\lambda_{ij} \tag{3.6}$$

Similarly, denoting the expected transition rate from j to $i \epsilon X^-$ by $E_j(t)$

$$E_j(t) = p_j(t) \sum_{i \in X^-} \lambda_{ji}$$

As $\Delta t \to 0^+$, $f_j(t)\Delta t$ and $E_j(t)\Delta t$ represent respectively the expected number of transitions into and out of state j in the interval $(t, t + \Delta t)$. Since the

probability of more than one transition in $(t, t + \Delta t)$ can be reasonably assumed as $0(\Delta t)$, the quantities $f_j(t)\Delta t$ and $E_j(t)\Delta t$ can be interpreted as the probabilities of a single transition into and out of state j. The difference $[f_j(t)\Delta t - E_j(t)\Delta t]$ therefore represents an increase in the probability of being in state j, i.e.

$$\Delta p_j(t) = f_j(t)\Delta t - E_j(t)\Delta t$$

As $\Delta t \to 0^+$

$$\frac{dp_j(t)}{dt} = -p_j(t) \sum_{i \in X^-} \lambda_{ji} + \sum_{i \in X^-} p_i(t)\lambda_{ij} \tag{3.7}$$

This can be recognized as the forward differential equation of state j. In matrix form

$$Ap(t) = p'(t) \tag{3.8}$$

where

$p(t)$ = The column matrix whose ith value $p_i(t)$ represents the probability of being in state i at time t for the given initial condition.
$p'(t)$ = The differential of $p(t)$
A = The transpose of the transition rate matrix used in the Markov approach.

The frequency balancing approach has been used to write the forward differential equations for the system and can be extended to derive expressions for the system reliability indices both in the time specific as well as the steady state domains.

Time Specific Domain

It is often necessary to examine the probable system performance over a finite interval of time. It is usual in the literature to define reliability indices in terms of system success or system failure. However, in many complex systems, there may be more than one degraded mode of failure. For example, a large chemical plant may not be just up or down but may have many possible capacity states. This is also true of transportation systems which may not be just available or unavailable and may have degrees of availability or unavailability. It is therefore appropriate to define reliability measures in terms of a subset X^+ where this subset may contain just one state or several states of the system. In particular applications the state will be referred to as success, failure, or some other appropriate name. In the transient domain the following indices are commonly used for repairable systems:

1 *Time Specific Availability of Subset* X^+

This is also designated in the literature as pointwise availability or instant availability and is the probability of the system being in any state contained in subset X^+ at a particular instant of time t. This will be denoted by $A_+(t)$ and since all states of the system are mutually exclusive, i.e. separated in time

$$A_+(t) = \sum_{i \in X^+} p_i(t) \tag{3.9}$$

The probability of being in state i at time t can be found from Equation (3.9) and has been discussed in Chapter 2. When X^+ is constituted by the system states which denote system failure, this can be called unavailability of the system at time t and designated by $U(t)$. This is probably the most widely used index in the transient domain.

2 *Fractional Duration of Subset* X^+

The fractional duration of subset X^+ in the interval (t_1, t_2) is also known as the interval availability or average time in X^+ and is defined as the expected proportion of the interval (t_1, t_2) spent in X^+ and denoted by $D_+(t_1, t_2)$. Since the states of the system are separated in time, the expected duration of X^+ is the sum of the expected durations in the states constituting X^+, i.e.

$$D_+(t_1, t_2) = \frac{1}{t_2 - t_1} \sum_{i \in X^+} \int_{t_1}^{t_2} p_i(t)dt = \frac{\int_{t_1}^{t_2} A_+(t)dt}{t_2 - t_1} \tag{3.10}$$

Equation (3.10) can be understood by considering probability as a relative duration. The probability of being in X^+ at time t, i.e. $A_+(t)$ can be considered constant over the interval $(t, t + \Delta t)$, as $\Delta t \to 0^+$. Since $A_+(t)$ can be considered constant over the interval, in a very large number of realizations of the associated stochastic process, on the average $A_+(t)\Delta t$ time will be spent in X^+ during $(t, t+\Delta t)$. Considering (t_1, t_2) to be divided into m increments of Δt

$$D_+(t_1, t_2) = \left[\lim_{m \to \infty} \sum_{m} A_+(t)\Delta t \right] / (t_2 - t_1)$$

$$= \frac{\int_{t_1}^{t_2} A_+(t)dt}{t_2 - t_1}$$

Another interpretation of Equation (3.10) can be provided by denoting the state occupied by t by $S_i(t)$ such that

$$S_i(t) = 1 \text{ if the system is in state } i$$
$$= 0 \text{ if otherwise.}$$

The proportion of (t_1, t_2) spent in X^+ is, therefore

$$T_+(t_1, t_2) = \frac{1}{t_2 - t_1} \sum_{i \in X^+} \int_{t_1}^{t_2} S_i(t)dt$$

where the right hand side can be regarded as the limit of a time average taken at points $(0, \Delta t, 2\Delta t, 3\Delta t, \ldots)$ as $\Delta t \to 0$. Therefore, the expected value $D_+(t_1, t_2) = E\{T_+(t_1, t_2)\}$

$$= \frac{1}{t_2 - t_1} \sum_{i \in X^+} \int_{t_1}^{t_2} E\{S_i(t)\}dt = \frac{1}{t_2 - t_1} \sum_{i \in X^+} \int_{t_1}^{t_2} P\{S_i(t) = 1\}dt$$

$$= \frac{1}{t_2 - t_1} \sum_{i \in X^+} \int_{t_1}^{t_2} p_i(t)\,dt = \frac{\int_{t_1}^{t_2} A_+(t)}{t_2 - t_1}$$

3 *Interval Frequency*

The interval frequency $F_+(t_1, t_2)$ is defined as the expected number of times the subset X^+ is encountered in the interval (t_1, t_2). Since the subset X^+ is said to have been encountered once, if the system transits from X^- to X^+, $F_+(t_1, t_2)$ represents the expected number of transitions from X^- to X^+ and not from X^+ to X^-. In this treatment the state transition rates are assumed constant which puts the stochastic process in the Markovian class. Similar treatments can, however, also be made for non-Markovian processes.

The interval frequency can be obtained by integrating the Expression (3.5) for time specific frequency over the interval, i.e.

$$F_+(t_1, t_2) = \int_{t_1}^{t_2} f_+(t)dt$$

$$= \sum_{i \in X^-} \int_{t_1}^{t_2} p_i(t) \sum_{j \in X^+} \lambda_{ij} dt \qquad (3.11)$$

Sometimes the interval frequency may be divided by the interval length to obtain average interval frequency.

Methods of Calculation

When the state space of the system is small, it may be possible to find explicit expressions for the interval frequency and fractional duration. When the state space becomes relatively large, this approach is not feasible. Methods for obtaining time specific probabilities have already been described in Chapter 2. This section extends those methods to the calculation of interval frequency and fraction duration.

Method 1

The time specific state probabilities can be found by solving the following differential equation in matrix form

$$P'(t) = P(t)R$$

with the initial condition $P(0) = I$, i.e. the identity matrix.

Here, $P(t)$ = The matrix whose (ij)th term $p_{ij}(t)$ denotes the probability of being in state j given that the process was in state i at $t = 0$

R = The transition rate matrix.

It was shown in the last chapter that if R has distinct eigenvalues then the probability matrix can be expressed as

$$P(t) = SD(t)S^{-1} \qquad (3.12)$$

where

$D(t)$ = The diagonal matrix whose (ii)th element is $\exp(r_i t)$, r_i being the ith eigenvalue of R

S, S^{-1} = The matrices formed from the right and left eigenvectors of R

If the distribution at $t = 0$ is given by the row vector $p(0)$, the distribution of t is given by

$$p(t) = p(0)P(t)$$

The ith element of the row vector $p(t)$ is denoted by $p_i(t)$ and represents the probability of being in state i at time t for the given condition at $t = 0$. After finding $p_i(t)$, $A_+(t)$ can be calculated using Equation (3.9).

Fractional duration

Since only $D(t)$ on the right hand side of (3.12) is time dependent

$$\int_{t_1}^{t_2} p(t)\,dt = p(0)S\int_{t_1}^{t_2} D(t)\,dt\,S^{-1}$$

$$= p(0)SM(t_1, t_2)S^{-1} \tag{3.13}$$

where

$$M(t_1, t_2) = \text{The diagonal matrix whose } (ii)^{\text{th}} \text{ term} = \frac{(e^{r_i t_2} - e^{r_i t_1})}{r_i}$$

Equation (3.13) thus yields $\int_{t_1}^{t_2} p_i(t)\,dt$ and by substituting these values in Equation (3.10), $D_+(t_1, t_2)$ can be calculated.

Interval Frequency

When the quantities $\int_{t_1}^{t_2} p_i(t)dt$ have been calculated by Equation (3.13), these can be substituted in (3.11) to determine $F_+(t_1, t_2)$

Method 2

If the initial state of the process is known, the differential equation becomes

$$p'(t) = p(t)R$$

The initial row vector $p(0)$ is such that its ith element $p_i(0)$ is equal to 1 if the process started in the ith state, otherwise it is 0. The process can be segmented into steps of a very small length Δt as shown in the last chapter and the probability vector at time $t = j\Delta t$ is

$$p(j\Delta t) = p(\overline{j-1}\,\Delta t)\,[I + R\Delta t] \tag{3.14}$$

The state probabilities can be calculated by repeated application of the above recursive relationship.

Fractional Duration

The fractional duration, using this technique, can be calculated using a discrete time equivalent of (3.10). Assuming $p_i(\Delta t)$ to be constant over the interval $(j\Delta t, (j+1)\Delta t)$

$$Dr(0, t) = \frac{1}{t}\sum_{j=0}^{m} p(j\Delta t)\Delta t \tag{3.15}$$

where

$Dr(0, t)$ = The row vector whose i^{th} entry, $D_i(0, t)$ represents the fractional duration of state i in the interval $(0, t)$.

Therefore

$$D_+(0, t) = \sum_{i \in X^+} D_i(0, t)$$

Interval Frequency

The interval frequency in the interval $(0,t)$ can be evaluated using a discrete time approximation of (3.11), i.e.

$$F_+(0, t) = t \sum_{i \in X^-} D_i(0, t) \sum_{j \in X^+} \lambda_{ij} \tag{3.16}$$

The above expression is readily understood by realizing that $t\,D_i(0,t)$ denotes the expected duration in state i, which when multiplied by the constant transition rate yields the expected number of transitions.

Steady State Domain

In many applications the time interval under consideration is very long, the stochastic process is remote from the time of origin and therefore the probability distribution has reached statistical equilibrium or is in a steady state condition. Under these conditions

$$\lim_{t \to \infty} p_i(t) = p_i$$

Therefore

$$\lim_{t \to \infty} E_{ij}(t) = E_{ij}$$

$$= \lambda_{ij} p_i$$

Equation (3.5) in this limiting condition becomes

$$f_+ = \sum_{i \in X^-} \sum_{j \in X^+} p_i \lambda_{ij} \tag{3.17}$$

Now as $t \to \infty$ Equation (3.7) becomes

$$0 = -p_j \sum_{i \in X^-} \lambda_{ji} + \sum_{i \in X^-} p_i \lambda_{ij}$$

That is

$$\begin{aligned} f_j &= \sum_{i \in X^-} p_i \lambda_{ij} \\ &= p_j \sum_{i \in X^-} \lambda_{ji} \\ &= E_j \end{aligned} \tag{3.18}$$

Equation (3.18) describes the frequency balance of state j with the rest of the state space. This means that the frequency of encountering state j is equal to the frequency of encountering the rest of the state space from state j. The frequency of encountering a state may therefore be computed either by calculating the expected transition rate out of the state or into the state. Purely from a conceptual viewpoint, however, the frequency of encountering a state is the expected transition rate into the state. This definition holds in both the transient as well as the steady state domain. If the subset X^+ consists of more than one state, then by following the same reasoning as for (3.7)

$$\sum_{i \in X^+} \frac{dp_i(t)}{dt} = -\sum_{i \in X^+} p_i(t) \sum_{j \in X^-} \lambda_{ij} + \sum_{j \in X^-} p_j(t) \sum_{i \in X^+} \lambda_{ji}$$

As $t \to \infty$

$$\begin{aligned} f_+ &= \sum_{j \in X^-} p_j \sum_{i \in X^+} \lambda_{ji} \\ &= \sum_{i \in X^+} p_i \sum_{j \in X^-} \lambda_{ij} \\ &= f_- \end{aligned} \tag{3.19}$$

In further treatment, the steady state frequency will be simply denoted as frequency. Equations (3.17) and (3.19) are of fundamental importance in determining the frequency of cumulative or individual states.

The Frequency, the Cycle Time and the Mean Duration

The probability of being in X^+ at any time in the interval $(t, t+T)$ as $t \to \infty$ is $p_+ = \sum_{i \in X^+} p_i$ and the corresponding frequency is f_+. Therefore, as $t \to \infty$, the interval frequency

$F_+(t, t+T)$ = The expected number of times X^+ is encountered in the interval $(t, t+T)$.

$$= Tf_+$$

Consequently

T^+ = The expected time between the two encounters of X^+, i.e., the mean cycle time

$$= T/F_+(t, t+T)$$

$$= 1/f_+ \tag{3.20}$$

Also

d_+ = The mean duration of X^+, i.e., the expected time of stay in X^+ in one cycle

$$= T^+ p_+ \tag{3.21}$$

$$= p_+/f_+ \tag{3.22}$$

Equations (3.19), (3.20), and (3.22) are the backbone of the frequency and duration method of system reliability evaluation. The application of the concepts discussed is now illustrated with the help of the following example.

Example: The state space diagram of two independent and identical components is shown below in Fig. 3.2. The failure and repair of each component is denoted by λ and μ respectively. The state description is as follows:

State 1	Both components up
State 2	One component is up and the other is down
State 3	Both components down

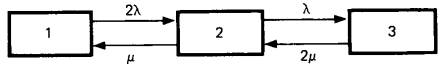

Fig. 3.2. State transition diagram of two independent identical components.

It is assumed that initially both the components are up. The calculation of various indices is now illustrated.

Time Specific Probabilities

The state differential equations can be written using the concept that the expected transition rate into the state minus the expected transition rate out of the state equals the rate of change of the state probability, i.e. using Equation (3.7)

$$\text{STATE 1:} \quad p_1'(t) = -2\lambda p_1(t) + \mu p_2(t)$$
$$\text{STATE 2:} \quad p_2'(t) = -(\mu + \lambda)p_2(t) + p_1(t)2\lambda + p_3(t)2\mu$$
$$\text{STATE 3:} \quad p_3'(t) = -2\mu p_3(t) + p_2(t)\lambda$$

The Laplace transform of the above equations, using the initial condition

$$p_1(0) = 1, \quad p_2(0) = 0, \quad \text{and} \quad p_3(0) = 0$$

is as follows

$$s\bar{p}_1(s) = 1 - 2\lambda\bar{p}_1(s) + \mu\bar{p}_2(s)$$
$$s\bar{p}_2(s) = -(\mu + \lambda)\bar{p}_2(s) + \bar{p}_1(s)2\lambda + \bar{p}_3(s)2\mu$$
$$s\bar{p}_3(s) = -2\mu\bar{p}_3(s) + \bar{p}_2(s)\lambda$$

Solving the above equations

$$\bar{p}_1(s) = \frac{1}{s+2\lambda} + \frac{2\lambda\mu(s+2\mu)}{s(s+\lambda+\mu)(s+2\lambda+2\mu)(s+2\lambda)}$$
$$\bar{p}_2(s) = \frac{2\lambda(s+2\mu)}{s(s+\lambda+\mu)(s+2\lambda+2\mu)}$$
$$\bar{p}_3(s) = \frac{2\lambda^2}{s(s+\lambda+\mu)(s+2\lambda+2\mu)}$$

Expanding into partial fractions

$$p_1(s) = \frac{1}{(\lambda+\mu)^2}\left[\frac{\mu^2}{s} + \frac{2\lambda\mu}{s+\lambda+\mu} + \frac{\lambda^2}{s+2(\lambda+\mu)}\right]$$
$$p_2(s) = \frac{2\lambda}{(\lambda+\mu)^2}\left[\frac{\mu}{s} + \frac{\lambda-\mu}{s+\lambda+\mu} - \frac{\lambda}{s+2(\lambda+\mu)}\right]$$
$$p_3(s) = \frac{\lambda^2}{(\lambda+\mu)^2}\left[\frac{1}{s} - \frac{2}{s+\lambda+\mu} + \frac{1}{s+2(\lambda+\mu)}\right]$$

Converting the above

$$p_1(t) = \frac{1}{(\lambda+\mu)^2}[\mu^2 + 2\lambda\mu e^{-(\lambda+\mu)t} + \lambda^2 e^{-2(\lambda+\mu)t}]$$
$$p_2(t) = \frac{2\lambda}{(\lambda+\mu)^2}[\mu + (\lambda-\mu)e^{-(\lambda+\mu)t} - \lambda e^{-2(\lambda+\mu)t}]$$
$$p_3(t) = \frac{\lambda^2}{(\lambda+\mu)^2}[1 - 2e^{-(\lambda+\mu)t} + e^{-2(\lambda+\mu)t}]$$

It should be appreciated that since the two components are statistically independent, the above equations could have more easily been derived by finding the probabilities of each component and using the product rule of probabilities.

Steady State Probabilities

As $t \to \infty$, the exponential terms disappear. The steady state values are

$$p_1 = \frac{\mu^2}{(\lambda+\mu)^2}$$
$$p_2 = \frac{2\lambda\mu}{(\lambda+\mu)^2}$$
$$p_3 = \frac{\lambda^2}{(\lambda+\mu)^2}$$

The remaining indices are calculated for state 3. The calculation for the other states is left to the reader as an exercise.

Fractional Duration

$$D_3(0,T) = \frac{1}{T}\int_0^T p_3(t)dt$$
$$= \frac{\lambda^2}{T(\lambda+\mu)^2}\left[T - 2\,\frac{1-e^{-(\lambda+\mu)T}}{\lambda+\mu} + \frac{1-e^{-2(\lambda+\mu)T}}{2(\lambda+\mu)}\right]$$

As $T \to \infty$

$$D_3(0,T) = \frac{\lambda^2}{(\lambda+\mu)^2}$$
$$= p_3$$

Interval frequency

$$F_3(0, T) = \int_0^T p_2(t)\lambda dt$$

$$= \frac{2\lambda^2}{(\lambda+\mu)^2}\left[\mu T + \frac{\lambda-\mu}{\lambda+\mu}(1-e^{-(\lambda+\mu)T}) - \frac{\lambda}{2(\lambda+\mu)}(1-e^{-2(\lambda+\mu)T})\right]$$

Steady State frequency

$$f_3 = p_3 2\mu$$

$$= \frac{2\lambda^2\mu}{(\lambda+\mu)^2}$$

It can be seen that as $T \to \infty$, the average interval frequency approaches the steady state frequency.

$$\lim_{T\to\infty} \frac{F_3(0,t)}{T} = \frac{2\lambda^2\mu}{(\lambda+\mu)^2}$$

$$= f_3$$

Mean Cycle Time to the Encounter of State 3.

$$T_3 = \frac{1}{f_3}$$

$$= \frac{(\lambda+\mu)^2}{2\lambda^2\mu}$$

Mean Duration of State 3.

$$d_3 = p_3 T_3$$

$$= \frac{1}{2\mu}$$

Frequency Equilibrium in a System of Independent Components

Equation (3.19) denotes the frequency equilibrium of the subset X^+ with the disjoint subset X^-. A case of special interest, from the point of view of the frequency equilibrium, is the system comprised of N independent binary components. The term 'binary components' is used here in the general sense of

a process which exists in either of the two states, up and down, the duration in each state being assumed exponentially distributed with the mean values of m and r. The reciprocal of m is λ which is the rate of transition from the up to the down state, and similarly μ, the reciprocal of r denotes the rate of transition from the down to the up state. In such a system it can be shown that in addition to frequency equilibrium of the state i with the disjoint subset containing the rest of the states, there is also a frequency equilibrium between any two individual states, i.e.

E_{ji} = The frequency encounter of state i from state j, i.e. the expected transitions per unit time from state j to state i.
= E_{ij}
= The frequency encounter of state j from state i.

This relationship is of considerable value when dealing with independent binary unit systems.

Proof: Let the state i comprise m components in the up state and k components in the down state. Whenever a component changes state, the system transits from state i to the state fitting the resulting description of component states. Let it be assumed that if the component '0' transits from the up to the down state, the system transits from state i to j. Then

$$E_{ij} = p_i\lambda_{ij}$$

$$= \left[\frac{\mu_0}{\mu_0+\lambda_0}\prod_{p\in A}\frac{\mu_p}{\mu_p+\lambda_p}\prod_{q\in B}\frac{\lambda_q}{\mu_q+\lambda_q}\right]\lambda_0 \qquad (3.23)$$

where A is a set containing all the components in the up state, except the component '0' which is in the up state but is not contained in the set A. Similarly B is a set containing all the components in the down state. Also

$$E_{ji} = p_j\lambda_{ji}$$

$$= \left[\frac{\lambda_0}{\mu_0+\lambda_0}\prod_{p\in A}\frac{\mu_p}{\mu_p+\lambda_p}\prod_{q\in B}\frac{\lambda_q}{\mu_q+\lambda_q}\right]\mu_0 \qquad (3.24)$$

From Equations (3.23) and (3.24)

$$E_{ij} = E_{ji}$$

Alternative Interpretation of Mean Cycle Time, Mean Duration and Mean Frequency

The emphasis in the preceding sections is on the frequency as an expectation of the state encounter rate. The mean cycle time is then obtained as the reciprocal of the mean frequency. This section describes an alternative approach by first deriving the expressions for mean times and then deducing the frequency relationships.

This treatment is intended for theoretically inclined readers and requires some acquaintance with renewal theory. This section may be omitted without any loss of continuity.

The entire state space is again assumed to be partitioned into two disjoint subsets X^+ and X^-.

Let

U = The random variable specifying the time of uninterpreted wandering of the system among the states $\{i \epsilon X^+\}$, i.e. starting in $i \epsilon X^+$, this is the time which the system spends in subset X^+ before once getting out.

D = The random variable defining the time of uninterrupted wandering of the system among the states $\{i \epsilon X^-\}$.

The sequence of random variables U and D defines an alternating renewal process. This is shown in Fig. 3.3. The random variable $U + D$ specifies the cycle time, i.e. the time between two successive encounters of subset X^+ or X^-. The quantity of interest is the mean value T_m of the random variable $U + D$, i.e. $E(U+D) = E(U) + E(D)$ where

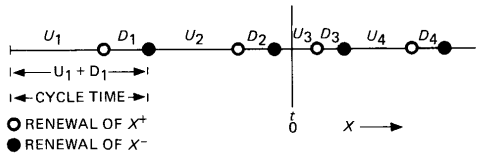

Fig. 3.3 The alternating renewal process (U,D) defined on the state space X.

$E(U)$ and $E(D)$ are respectively the mean values T_u, T_d of the random variables U and D. In long term or steady state analysis the interval of interest is $(t, t + T)$, $t \to \infty^+$. If the origin is, however, fixed at t, i.e. the time is represented by x such that $x = 0$, at t, then the origin of the process is at $t \to -\infty$, i.e. the process started remote from the time origin. Such a process is called an equilibrium alternating renewal process.

Let

$$S_U(u) = \text{The survival function of } U$$
$$= P\{U > u\}$$

For the equilibrium renewal process, the residual life time of random variable U, i.e. the time y from a random instant $t \epsilon U$ (i.e. under the condition that the system is in subset X^+ at the instant t) to the termination of U has the probability density function

$$f^e_{U_r}(x) = \frac{S_u(x)}{T_u}$$

The survival function $S_{U_r}(x)$ of the residual life time of U, U_r therefore is

$$S_{U_r}(x) = \frac{1}{T_u} \int_x^\infty S_U(u) du$$

Differentiating this expression

$$\frac{dS_{U_r}(x)}{dx} = -\frac{1}{T_u} S_U(x)$$

Since $S_U(0) = 1$

$$T_u = \left[-\frac{dS_{U_r}(x)}{dx} \bigg|_{x=0} \right]^{-1} \tag{3.25}$$

Now

$$S_{U_r}(x) = P\{U_r > x\}$$
$$= \sum_{i \in X^+} P_i^* r_i(x) \tag{3.26}$$

where

P_i^* = $P\{Z = i \,|\, i \in X^+\}$, i.e., the conditional steady state probability that the system is in the ith state given that it is in the subset X^+

$Z = \underset{t \to \infty}{Z(t)}$

and

$r_i(x)$ = The probability that the system which begins to operate in state $i \in X^+$ at some random instant of time in the equilibrium process will not once get out of X^+ during x.

It is obvious that

$$P_i^* = p_i \left(\sum_{i \in X^+} p_i \right)^{-1} \tag{3.27}$$

Since the transition rates λ_{ik} are assumed constant, the transitions from state i to state k are governed by random variables which have exponential probability density functions. The probability density and the survival functions of the kth process will be

$$f_{ik}(x) = \lambda_{ik} e^{-\lambda_{ik} x}$$

and

$$S_{ik}(x) = e^{-\lambda_{ik} x}$$

When the probability density function is exponential, the instant of entry does not affect the probability density function, i.e. the probability density function of the residual life time of the random variable is the same as that of the random variable. Therefore, the probability that the system which begins to operate in the state $i \epsilon X^+$ at some random instant of time, will not once get out of state i to state j

$$\begin{aligned} \eta_{ij}(x) &= \int_x^\infty f_{ij}(u) \prod_{\substack{k=1 \\ k \neq i,j}}^{n} S_{ik}(u) du \\ &= \int_x^\infty \lambda_{ij} \exp \left\{ - \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} u \right\} du \\ &= \lambda_{ij} \left(\sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} \right)^{-1} \exp \left\{ - \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} x \right\} \end{aligned}$$

Therefore

$$\begin{aligned} r_i(x) &= \sum_{j \in X^-} \eta_{ij}(x) + \sum_{j \in X^+} \int_0^x (1 - Q_j(y)) d\eta_{ij}(x - y) \\ &= \sum_{j \in X^-} \lambda_{ij} \left(\sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} \right)^{-1} \exp \left\{ - \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} x \right\} \\ &\quad + \sum_{j \in X^+} \lambda_{ij} \exp \left\{ - \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} x \right\} \int_0^x (1 - Q_j(y)) \exp \left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} y \right\} dy \end{aligned}$$

where $Q_j(y)$ is the probability distribution of the time at which the system gets out of X^+ given that the system entered state $j \epsilon X^+$ at $y = 0$.

Substituting into (3.26)

$$\begin{aligned} S_{U_r}(x) &= \sum_{i \in X^+} P_i^* \left[\sum_{j \in X^-} \lambda_{ij} \left(\sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} \right)^{-1} \exp \left\{ - \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} x \right\} \right. \\ &\quad + \sum_{j \in X^+} \lambda_{ij} \exp \left\{ - \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} x \right\} \int_0^x (1 - Q_j(y)) \\ &\quad \left. \times \exp \left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} y \right\} dy \right] \end{aligned}$$

$$\begin{aligned} \frac{dS_{U_r}(x)}{dx} &= \sum_{i \in X^+} P_i^* \left[\left(- \sum_{j \in X^-} \lambda_{ij} \exp \left\{ - \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} x \right\} \right) \right. \\ &\quad + \sum_{j \in X^+} \left(\lambda_{ij} \exp \left\{ - \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} x \right\} ((1 - Q_j(x)) \exp \left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} x \right\} \right) \\ &\quad - 1 - \lambda_{ij} \left(\sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} x \exp \left\{ - \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik} x \right\} \right. \\ &\quad \left. \left. \times \int_0^x (1 - Q_j(y)) \exp \left\{ \sum_{\substack{k=1 \\ k \neq i}}^{n} \lambda_{ik}(y) \right\} dy \right) \right] \end{aligned}$$

and

$$\left. \frac{dS_{U_r}(x)}{dx} \right|_{x=0} = - \sum_{i \in X^+} P_i^* \sum_{j \in X^-} \lambda_{ij}$$

Substituting into (3.25) and substituting the value of P_i^* from (3.27)

$$T_u = \sum_{i \in X^+} p_i \left(\sum_{i \in X^+} p_i \sum_{j \in X^-} \lambda_{ij} \right)^{-1} \tag{3.28}$$

Similarly

$$T_d = \sum_{i \in X^-} p_i \left(\sum_{i \in X^-} p_i \sum_{j \in X^+} \lambda_{ij} \right)^{-1} \tag{3.29}$$

The expressions for T_u and T_d have been derived by examining the distribution of the states constituting X^+ and X^- under stationary conditions. The behaviour of the alternating renewal process $\{U, D\}$ is now examined in detail. Since $x = 0$ is at $t \to \infty^+$ (see Fig. 3.3)

$P_u(x=0)$ = The probability that the system is in X^+ at the observation origin, given the process started a long time ago

$$= P_U(t \to \infty^+)$$

$$= \sum_{i \in X^+} p_i$$

$$= \frac{T_u}{T_u + T_d} = P_U(x) \tag{3.30}$$

Similarly

$$P_D(x=0) = P_D(x) = \sum_{i \in X^-} p_i = \frac{T_d}{T_u + T_d} \tag{3.31}$$

Substituting (3.30) and (3.31) into (3.28) and (3.29) respectively

$$T_m = T_u + T_d = \left(\sum_{i \in X^+} P_i \sum_{j \in X^-} \lambda_{ij} \right)^{-1}$$

$$= \left(\sum_{i \in X^-} p_i \sum_{j \in X^+} \lambda_{ij} \right)^{-1} \tag{3.32}$$

This expression for the cycle time is the same as that derived using the expectation concept. The frequency of encountering X^+ at x is the same as the renewal density of the random variable $U + D$. The renewal density is defined as

$$h(x) = \frac{E(N_{x,x+\Delta x})}{\Delta x \atop \Delta x \to 0^+}$$

where

$N_{x,x+\Delta x}$ = The random variable representing the number of renewals in $(x, x + \Delta x)$

E Denotes the expectation

Now, let

$h_{ij}(x)$ = The renewal density of subset j given that the system is in i at $x = 0$

It can be shown that for a modified renewal process, i.e. a sequence of independent random variables in which all the random variables are identically distributed except the first one which has a different distribution, the expression for renewal density is

$$\bar{h}^{(m)}(s) = \frac{\bar{f}_1(s)}{1 - \bar{f}(s)}$$

where

$\bar{f}_1(s)$, $\bar{f}(s)$ and $\bar{h}^{(m)}(s)$ are the Laplace transforms of $f_1(x)$, $f(x)$ and $h^{(m)}(x)$ respectively.

The quantities $f_1(x)$ and $f(x)$ are the probability density functions of the first and the subsequent random variables respectively.

1. Determination of $\bar{h}_{X^+X^-}(s)$

If the system is assumed to be in subset X^+ at 0, the first renewal of X^- or the encounter of X^+ occurs at $U_1 + D_1$ where the subscript indicates the number determined from $x = 0$ and not $t = 0$ (see Fig. 3.3) and the prime indicates that the distribution of the first U is different from the subsequent ones. The second renewal of X^- is at $U_2 + D_2$ and so on.

Now

$$f_{U_1'}(x) = \frac{S_U(x)}{T_u}$$

and

$$f_{D_1}(x) = f_D(x)$$

Therefore

$$\bar{f}_1(s) = \frac{(1 - \bar{f}_U(s))\bar{f}_D(s)}{T_u . s}$$

and

$$\bar{f}(s) = \bar{f}_U(s)\bar{f}_D(s)$$

Therefore

$$\bar{h}_{X^+X^-}(s) = \frac{(1 - \bar{f}_U(s))\bar{f}_D(s)}{T_u s(1 - \bar{f}_U(s)\bar{f}_D(s))} \tag{3.33}$$

2. Determination of $\bar{h}_{X^-X^-}(s)$

$$\bar{f}_1(s) = \frac{1-\bar{f}_D(s)}{T_d s}$$

and

$$\bar{f}(s) = \bar{f}_U(s)\bar{f}_D(s)$$

Therefore

$$\bar{h}_{X^-X^-}(s) = \frac{1-\bar{f}_D(s)}{T_d s(1-\bar{f}_U(s)\bar{f}_D(s))} \tag{3.34}$$

Now

$$f_{+}(x) = h_{X^-}(x) = \bar{h}_{X^+X^-}(x).P\{Z(0)=i\in X^+\} + h_{X^-X^-}(x).P\{Z(0)=i\in X^-\}$$

i.e.

$$\bar{f}_{+}(s) = \bar{h}_{X^-}(s) = \bar{h}_{X^+X^-}(s).P^+ + \bar{h}_{X^-X^-}(s)P^-$$

$$= \frac{1}{s(T_u+T_d)}$$

Converting

$$f_{+}(x) = f_{+} = \frac{1}{T_u+T_d}$$

$$= \frac{1}{T_m}$$

$$= \sum_{i\in X^-} p_i \sum_{i\in X^+} \lambda_{ij}$$

$$= \sum_{i\in X^+} p_i \sum_{j\in X^-} \lambda_{ij}$$

$$= f_{-}(x) = f_{-} \tag{3.35}$$

since it can be proved similarly that

$$f_{-}(x) = f_{-} = \frac{1}{T_m}$$

Also from (3.29)

$$T_u = \text{The mean duration in } X^+$$

$$= T_m P_+$$

$$= P_+/f_+ \tag{3.36}$$

It can, therefore, be seen that the expressions for frequency, mean cycle time and mean duration are the same as those derived using the expectation concept.

The Relationship to Average Values

It is quite well known that the arithmetic mean of a variable tends to its expected value as the number of trials becomes large. This section examines its application to the cycle time, the mean duration and the frequency indices. It can be seen from Fig. 3.3 that the cycle time between two successive encounters of X^+ is characterized by the random variable $T = U + D$. Thus $T_1 = U_1 + D_1$ is the cycle time to the first encounter of X^+ and $T_i = U_i + D_i$ is the cycle time between the $(i\text{-}1)$th and the ith encounter of X^+. The random variables T_i are independently and identically distributed with mean T_m. Assume the random variable T_i to be observed n times and define the random variable

$$\bar{T} = \frac{T_1 + T_2 + \ldots + T_n}{n}$$

Then for any constant $\epsilon > 0$

$$\lim_{n\to\infty} P\{|\bar{T} - T_m| > \epsilon\} = 0$$

This can be interpreted that, as the number of encounters of X^+ increases, the average cycle time approaches the mean cycle time with a probability of one. The mean cycle time found by Equation (3.32) is therefore the long run average interval between the two successive encounters of X^+. Similarly it can be seen that T_u and T_d are the long run average residence periods of the system in X^+ and X^-. Since the frequency is the reciprocal of the mean cycle time, it can be appreciated that it is also a long term average.

The Concept of Equivalent Transition Rate

The concept of equivalent transition rate plays an important role in system reliability evaluation. Equation (3.5) can be written as

$$f_{+}(t) = \sum_{i\in X^-} \sum_{j\in X^+} p_i(t)\lambda_{ij}$$

$$= P_{-}\lambda^{(e)}_{X^-X^+}(t)$$

where

$$P_{-} = \sum_{i\in X^-} p_i(t)$$

and

$\lambda^{(e)}_{X^-X^+}(t)$ = The equivalent transition rate from subset X^- to X^+.

Therefore

$$\lambda^{(e)}_{X^-X^+}(t) = \sum_{i \in X^-} \sum_{j \in X^+} \lambda_{ij} \Big/ \sum_{i \in X^-} p_i(t) \tag{3.37}$$

The most important application of this concept is in reducing the system state space. The states can be merged and the equivalent transition rate from the merged states found by the application of Equation (3.37). The full implications of the equivalent transfer rate and the limitations on its use are outlined in Chapter 5 while deriving conditions of mergeability.

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