Module 4-3
Methods of Quantitative Reliability Analysis

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METHODS OF QUANTITATIVE RELIABILITY ANALYSIS

- ANALYTICAL METHODS
  - STATE SPACE USING MARKOV PROCESSES
  - NETWORK REDUCTION
  - MIN CUT SETS

- MONTE CARLO SIMULATION
  - NONSEQUENTAL - RANDOM SAMPLING
  - TIME SEQUENTIAL

- CONCEPT OF RELIABILITY COHERENCE

Monte Carlo Simulation

- Introduction

  - The Monte Carlo method mimics the failure and repair history of components and the system by using the probability distributions of component states.
  - Statistics are collected and indices estimated by statistical inference.
  - Two main approaches: random sampling , sequential simulation.

Random number generator

- Each number should have equal probability of taking on any one of the possible values and it must be statistically independent of the other numbers in the sequence. Random numbers in a given range follow a uniform probability density function.

Multiplicative congruential method:

Random number R_{n+1}, can be obtained from R_n :

$$R_{n+1} = (aR_n)(\text{modulo } m)$$

where

a,m= positive integers, a < m.

R_{n+1} is the remainder when $(a R_n)$ is divided by m.

Random number generator

suggested values

a=455 470 314

m=2^{31} -1 = 2147 483 647

R0 = seed

= any integer between 1 and 2147 483 64

- Range can be limited by truncation.
  - If rn between 0 and 999 are required, the last three digits of the random number generated can be picked up.

Random sampling

- Sampling a component state:

Consider a component that has probability distribution:

State number (random variable)	Probability
1	.1
2	.2
3	.4
4	.2
5	.1

Let us assume that the random numbers lie in the range 0 to 1. We can assign the random numbers proportional to their probability as follows:

Random number drawn	State sampled
0 to.1	1
.1+ to .3	2
.3+ to .7	3
.7+ to .9	4
.9+ to 1.	5

Random sampling

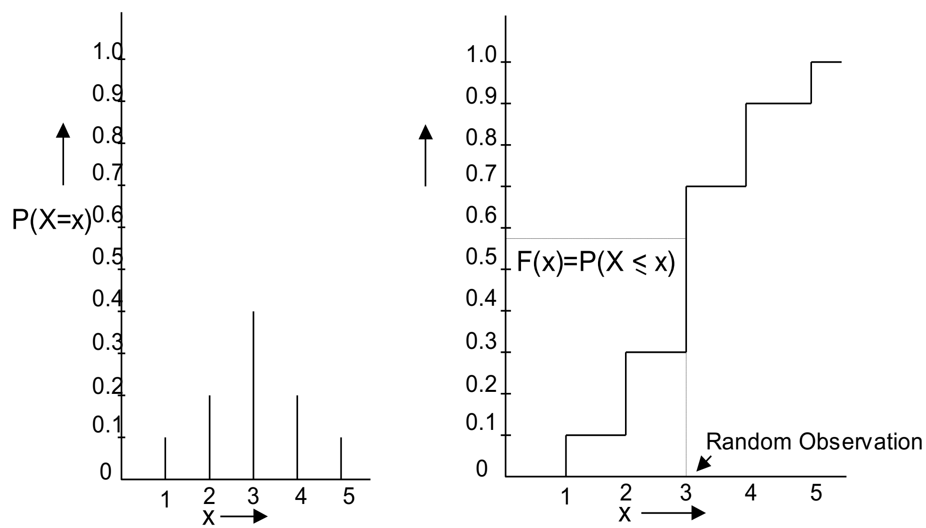
State number (random variable)	Probability
1	.1
2	.2
3	.4
4	.2
5	.1

Random number drawn	State sampled
0 to.1	1
.1+ to .3	2
.3+ to .7	3
.7+ to .9	4
.9+ to 1.	5

So if a rn is .56 ,then we say that state number 3 is sampled.

Random sampling

- This procedure can be more simply carried out by using a cum prob or probability distribution function. The prob mass function and corresponding probability distribution function for this component are shown below



- Now you can place the rn on the vertical axis and read the value of the state sampled on the horizontal axis. It can be seen that this is equivalent to proportional sampling.

Random sampling

- Sampling a system state:

- If a system consists of n independent components, then to sample a system state, n random numbers will be needed to sample the state of each component.
- For example for a system of two components with the pdfs shown above, sampling may proceed as follows. Random numbers used are found by a computer program and are shown on the next page.

Random sampling

RN for component 1	RN for comp 2	System state
.946	.601	(5,3)
.655	.671	(3,3)
.791	.333	(4,3)
.345	.532	(3,3)
.438	.087	(3,1)
.311	.693	(3,3)
.333	.918	(3,5)
.998	.209	(5,2)
.923	.883	(5,4)
.851	.135	(4,2)
.651	.034	(3,1)
.316	.525	(3,3)
.965	.427	(5,3)
.839	.434	(4,3)

Now if you want to estimate the probability of state (3,3)

P(3,3) = n/N

where

n = number of times state sampled

N = total number of samples.

From the table

P(3,3) = 4/14

=2/7

=.286

The actual prob of (3,3) is

.4x.4 = .16

If this sampling and estimation are continued the estimated value will come close to .16.

Random sampling

- The problems of Monte Carlo is that the indices obtained are estimates.
- So one must have some criterion to decide whether the indices have converged or not.

A Sequence of Random Numbers Generated

.946	.308
.601	.116
.655 .671	.864 .917
.791 .332	.126 .157
.345 .532	.837 .029
.438 .087	.865 .603
.311 .693	.203 .823
.333 .918	.969 .509
.998 .209	.407 .355
.923 .883	.885 .931
.851 .135	.896 .680
.651 .034	.057 .922
.316 .525	.113 .852
.965 .427	.931 .538
.839 .434	.504 .925

Sequential Simulation

- Sequential simulation can be performed either by advancing time in fixed steps or by advancing to the next event.

- Fixed time interval method:
- This method is useful when using Markov chains,i.e., when transition probabilities over a time step are defined.
- This is simulated using basically the proportionate allocation technique described under sampling.
- This is indeed sampling conditioned on a given state.

Sequential Simulation

- Fixed time interval method

This can be illustrated by taking an example. Assume a two state system and the probability of transiting from state to state over a single step given by the following matrix:

Initial State	Final State	
	0	1
0	0.3	0.7
1	0.4	0.6

Starting in state 0, select a rn and the next state is determined as follows:

digit	event
0 to 0.3	stay in state 0
0.3+ to 1.0	transit to 1

Similarly starting in state 1,

digit	event
0 to 0.4	transit to 0
0.4+ to 1.0	stay in 1

Sequential Simulation

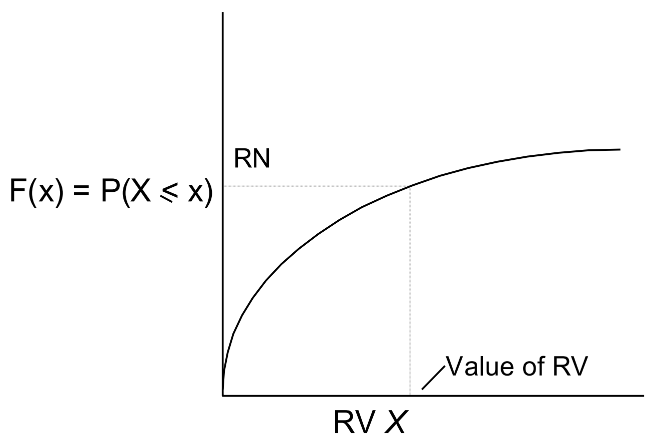
- Construction of a realization for 10 steps:

Step	RN	State
1	.947	0
2	.601	1
3	.655	1
4	.671	1
5	.791	1
6	.333	1
7	.345	0
8	.531	1
9	.478	1
10	.087	1

Sequential Simulation

- Next event method
- This method is useful when the times in system states are defined using continuous variables with pdfs.
- Keeping in mind that discrete rvs are only a special case of continuous rvs ,the previously described method of reading a variable from the pdf using a rn can be used.
- Consider for example a rv X representing the up time of a component and distribution as shown in the following figure.
- Now if a rn between 0 and 1. is drawn, then the time to failure can be read as shown:

$$F(x) = P(X \leq x)$$



Sequential Simulation

- Continuous distributions are approximated by discrete distributions whose irregularly spaced points have equal probabilities.
- The accuracy can be increased by increasing the number of intervals into which (0,1) is divided. This requires additional data in the form of tables.
- Although the method is quite general, its disadvantages are the great amount of work required to develop tables and possible computer storage problems.
- The following analytic inversion approach is simpler.

Sequential Simulation

Let Z be a random number in the range 0 to 1 with a uniform probability density function, i.e., a triangular distribution function:

$$f(z) = \begin{cases} 0 & Z < 0 \\ 1 & 0 \le Z \le 1 \\ 0 & Z > 0 \end{cases}$$

Similarly

$$F(z) = \begin{cases} 0 & Z < 0 \\ z & 0 \le Z \le 1 \\ 1 & Z > 1 \end{cases}$$

Sequential Simulation

Let F(x) be the distribution from which the random observations are to be generated. Let

$$z = F(x)$$

Solving the equation for x gives a random observation of X. That the observations so generated do have F(x) as the probability distribution can be shown as follows.

Let ϕ be the inverse of F; then

$$x = \phi(z)$$

Now x is the random observation generated. We determine its probability distribution as follows

$$\begin{aligned} P(x \le X) &= P(F(x) \le F(X)) \\ &= P(z \le F(X)) \\ &= F(X) \end{aligned}$$

Therefore the distribution function of x is F(X), as required.

- In the case of several important distributions, special techniques have been developed for efficient random sampling.

Sequential Simulation

- In most studies, the distributions assumed for up and down times are exponential. The exponential distribution has the following probability distribution

$$P(X \leq x) = 1 - e^{-\rho x}$$

- where **1/ρ** is the mean of the random variable X. Setting this function equal to a random decimal number between 0 and 1,

$$z = 1 - e^{-\rho x}$$

- Since the complement of such a random number is also a random number, the above equation can as well be written as

$$z = e^{-\rho x}$$

Sequential Simulation

- Taking the natural logarithm of both sides and simplifying, we get

$$x = -\frac{\ln(z)}{\rho}$$

- which is the desired random observation from the exponential distribution having **1/**ρ as the mean.

Sequential Simulation

- This method is used to determine the time to the next transition for every component, using λ or μ for ρ, depending on whether the component is UP or DOWN.
- The smallest of these times indicates the most imminent event, and the corresponding component is assigned a change of state. If this event also results in a change of status, (e.g., failure or restoration) of the system, then the corresponding system indices are updated.

Sequential Simulation

- The procedure can be illustrated using an example of two components with data as given below.

Component	λ (f/hr)	μ (rep/hr)
1	.01	.1
2	.005	.1

Sequential Simulation

- → indicates time causing change and is added to obtain the total time.

Time	Random Number for Component		Time to change		Component state	
	1	2	1	2	1	2
0	.946	.601	5 ←	101	U	U
5	.655	-	0/4 ←	96	D	U
9	.670	-	0/40 ←	92	U	U
49	.790	-	0/2 ←	52	D	U
51	.332	-	0/110	50 ←	U	U
101	-	.345	60	0/11 ←	U	D
112	-	.531	49 ←	0/127	U	U
161	.437	-	0/8 ←	78	D	U
169	.087	-	0/244	70 ←	U	U
239	-	.311	174	0/12 ←	U	D
251	-	.693	162	0/73 ←	U	U
324	-	.333	89	0/	U	D
		↓				

Estimation and Convergence in Sampling

- It is crucial to sample sufficient number of states to estimate reliability indexes.
- This section describes the estimation and convergence criterion of both techniques
  - random sampling
  - sequential sampling.

Random Sampling

In random sampling, $x \in X$ are sampled from their joint distributions. Then estimate of $E[F(x)]$ is

$$\hat{E}[F(x)] = \frac{1}{N_S} \sum_{i=1}^{N_S} F(x_i)$$

where

N_S = Number of sampling
$F(x_i)$ = Test result for i^{th} sampled value

And the variance of this estimate is,

$$Var(\hat{E}[F(x)]) = \frac{Var(F(x))}{N_S}$$

Since $Var(F(x))$ is not known, its estimate can be used as

$$\hat{Var}(F(x)) = \frac{1}{N_S} \sum_{i=1}^{N_S} (F(x_i) - \hat{E}[F(x)])^2$$

Random Sampling

Convergence is typically based on the value of the coefficient of variation (COV),

$$COV = \frac{SD(\hat{E}[F(x)])}{\hat{E}[F(x)]}$$

where

$SD(\hat{E}[F(x)])$ = Standard deviation of the estimator of $E[F(x)]$

Since $SD(\hat{E}[F(x)]) = \sqrt{Var(\hat{E}[F(x)])}$,

$$COV = \frac{1}{\hat{E}[F(x)]}\sqrt{\frac{Var(F(x))}{N_S}}.$$

Hence,

$$N_S = \frac{Var(F(x)}{(COV \times \hat{E}[F(x)])^2}.$$

Random Sampling

$$N_S = \frac{Var(F(x)}{(COV \times \hat{E}[F(x)])^2}.$$

It can be seen

1. Sample size is not affected by system size or complexity.
2. Accuracy required and the probability being estimated effect the sample size.
3. Computational effort depends on NS and CPU time/sample.

Sequential Sampling

In general terms, states of the system are generated sequentially by transition from one state to the next using probability distributions of component state durations and random numbers from [0,1].

Consider a component i, assume that this component is up and duration of up state is given by U_i (a random variable). If Z is a random number then the observation of up time can be drawn

$$z = Pr(U_i \leq U) = F_i(U)$$

The algorithm can be described in the following steps. Let us assume that the n^{th} transition has just taken place at time t_n and the time to next transition of component i is given by T_i. Thus the vector of times to component transitions is given by T_i and the simulation proceeds in the following steps.

Sequential Sampling

Step 1. The time to next system transition is

$$T = \min T_i$$

If this T corresponds to T_p , that is, the p^{th} component, then next transition takes place by the change of state of this component.

Step 2. The simulation time is now advanced,

$$t_{n+1} = t_n + T$$

Step 3. The residual times to component transitions are calculated

$$T_i^r = T_i - T$$

where T_i^r = Residual time to transition of component i

Sequential Sampling

Step 4. The residual time for component p causing transition becomes zero and time to its next transition T_p is determined by drawing a random number.
Step 5. The time T_i is set as shown

$$T_i = \begin{cases} T_i^r & i \neq p \\ T_p & i = p \end{cases}$$

Step 6. From t_n to t_{n+1} ,the status of equipment stays fixed and the following steps are performed.

1. The load for each node is updated to the current hour.
2. If no node has loss of load, the simulation proceeds to the next hour otherwise state evaluation module is called.
3. If after remedial action all loads are satisfied, then simulation proceeds to next hour. Otherwise, this is counted as loss of load hour for those nodes and the system. Also if in the previous hour there was no loss of load, then this is counted as one event of loss of load.
4. Steps (a) - (c) are performed until t_{n+1}.

Sequential Sampling

Step 7. The statistics are updated as described after Step 8 and the process moves to Step 2.

Step 8. The simulation is continued until convergence criterion is satisfied. Let,

I_i = Value of reliability index (for example number of hours of load loss) obtained from simulation data for year i
N_y = Number of years of simulated data available
SD_I = Standard deviation of the estimate I

Then, estimate of the expected value of the index I is

$$\hat{I} = \frac{1}{N_y} \sum_{i=1}^{N_y} I_i$$

Variance of the estimate is

$$Var(\hat{I}) = \frac{\sum_{i=1}^{N_y} (I_i - \hat{I})^2}{{N_y}^2}$$

COV is

$$COV = \frac{\sqrt{Var(\hat{I})}}{\hat{I}}$$

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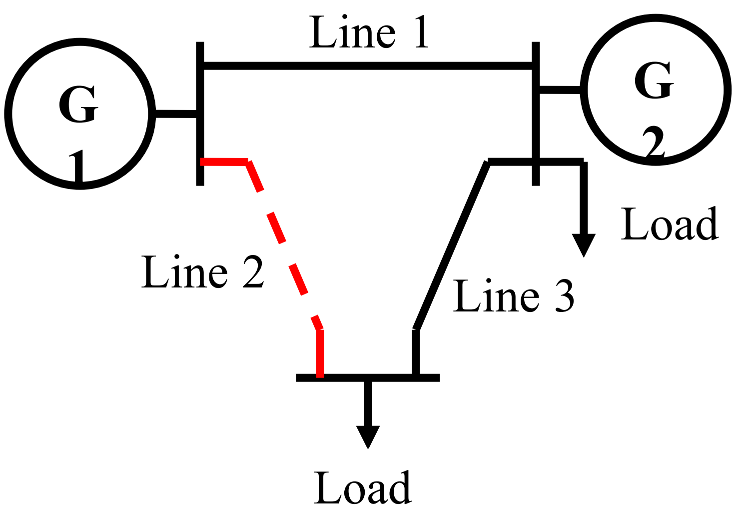
- CONCEPT OF RELIABILITY COHERENCE

Concept of Reliability Coherence

- If the system is failed with some components healthy and some components failed, then for a reliability coherent system, failure of a healthy component will not lead to system success.
- If the system is in success state with some components healthy and some components failed, then for a reliability coherent system, repair/restoration of a failed component will not lead to system failure.

System Status	Failed Components	Healthy Components
Success	Repair of these components will not lead to system failure	-
Failure	-	Failure of these components will not lead to system success

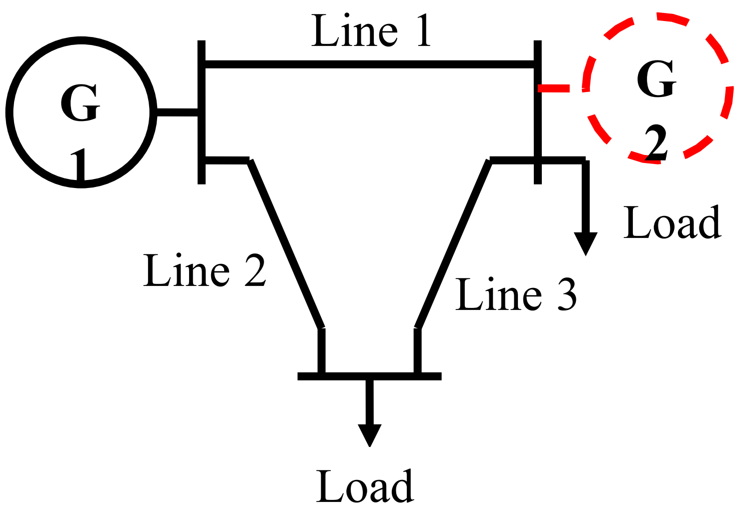
Example 1



Assumed System Status	Failed Components	Healthy Components
Success	Line 2	G1, G2, Line 1 and Line 3

- Reliability Coherence: Repair of Line 2 will not lead to system failure

Example 2



Assumed System Status	Failed Components	Healthy Components
Failure	G2	G1, Line 1, Line 2 and Line 3

- Reliability Coherence: Failure of Line 2 will not lead to system success

REFERENCES

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3. R. Billinton, R. Allen, “Reliability Evaluation of Power Systems”, Plenum Press, 1984