Module 4-1
Methods of Quantitative Reliability Analysis

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METHODS OF QUANTITATIVE RELIABILITY ANALYSIS

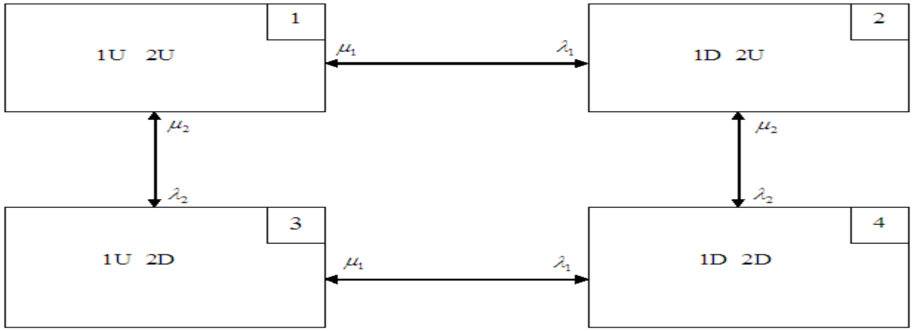
- ANALYTICAL METHODS
  - STATE SPACE USING MARKOV PROCESSES
  - NETWORK REDUCTION
  - MIN CUT SETS
- MONTE CARLO SIMULATION
  - NONSEQUENTAL - RANDOM SAMPLING
  - TIME SEQUENTIAL
- CONCEPT OF RELIABILITY COHERENCE

State Space Approach

- Identify all possible states
- Determine interstate transition rates
- Calculate state probabilities
- Calculate reliability indices

Formulation of transition rate matrix

- Identify system states resulting from failure and repair of components and other changes.
- The interstate transition rates are next determined and may be shown in a state transition diagram.
- The state transition diagram of a 2-component system is shown in Fig. a, as an example.



$\lambda_i, \mu_i =$ Failure and repair rates of component i

Fig a. State transition diagram of a 2 component system

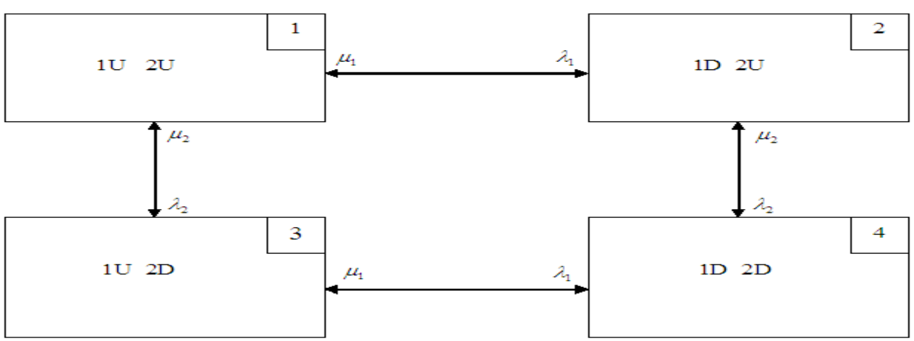
Formulation of transition rate matrix

- The state transition diagram is a helpful visual aid but for computational purposes this information is arranged in the form of a transition rate matrix.
- The element of matrix A are such that

$$a_{ij} = \lambda_{ij}$$

$$i \neq j \qquad\qquad \lambda_{ij} = \text{transition rate from state } i \text{ to state } j$$

$$a_{ii} = -\sum_{j} \lambda_{ij}$$



$$A = \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 \\ \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 \\ \mu_2 & 0 & -(\mu_2+\lambda_1) & \lambda_1 \\ 0 & \mu_2 & \mu_1 & -(\mu_1+\mu_2) \end{bmatrix}$$

Calculation of State Probabilities

- In general, the steady state probabilities can be obtained by solving equation BP = C

B = Matrix obtained from A' by replacing the elements of an arbitarily selected row k by 1

A' = Transpose of transition rate matrix:

$$a_{ij} = \lambda_{ji} \text{ and } a_{ii} = -\sum_j \lambda_{ji}$$

λ_{ij} = Constant transition rate from state i to state j

P = Column vector whose ith term p_i is the steady state probability of the system being in state i

C = Column vector with th kth element equal to 1 and other elemnts set to zero

- If the components are independent, the state probabilities can be obtained more simply using multiplication rule.

Calculation of State Probabilities

- **System Unavailability**: Steady state probability of the system being in the failed state or unacceptable states. It is also defined as the long run fraction of the time that system spends in the failed state.

$$P_f = \sum_{i \in F} p_i$$

Where

P_f = System unavailability or probability of system failure

F = Subset of failed states

- **Frequency of System Failure**: The expected number of failures per unit time (e.g. per year).

$$f_f = \sum_{i \in (S-F)} p_i \sum_{j \in F} \lambda_{ij} = \sum_{i \in F} p_i \sum_{j \in (S-F)} \lambda_{ij}$$

where f_f = frequency of system failure

S = system state space

Calculation of State Probabilities

- **Mean cycle time:** The expected time between successive failures

$$T_f = 1/f_f$$

- **Mean down time:** The mean time spent in the failed state during each system failure event. In other words this is the expected time of stay in F in one cycle of system up and down periods.

$$d_f = P_f / f_f$$

- **Mean up time: T**he mean time of stay in the system up state before system failure.

$$d_u = T_f - d_f$$

- **Two basic indices of system reliability are the probability and frequency of failure. The other indices can be computed from these two indices.**

Matrix Approach
To Index Calculations

$$f_f = U\bar{A}Q,$$

Where

U = Row vector having number of elements equal to the number of system states, the ith element of U,

$$u_i = 1 \text{ if state } i \in F$$
$$= 0 \text{ otherwise}$$

Q = Column derived from P by substituting $p_i = 0$ if state $i \in F$

and

$\bar{A}$ = A′ with diagonal elements set to zero.

Matrix Approach
To Index Calculations

An alternative approach

$$f_f = \overline{U}\overline{A}\overline{Q},$$

where

$$\overline{U} = \text{Row vector: the } ith \text{ element of } \overline{U},$$

$$u_i = 0 \text{ if state } i \in F$$

$$= 1 \text{ otherwise}$$

$$\overline{Q} = \text{Column vector derived from P by substituting}$$

$$p_i = 0 \text{ if state } i \in (S - F)$$

Sequential Model Building

- Development of a transition rate diagram or matrix for a large system in one step may sometimes be impractical.
- Alternatively, the state transition diagrams can be developed for subsystems, reduced by merging similar states and then the reduced diagrams can be combined to yield the system state transition diagram.
- When the states are combined, the equivalent transition between the subsets of merged states can be obtained using the following equation.

$$\lambda^{e}_{XY} = \left(\sum_{i \in X} \sum_{j \in Y} p_i \, \lambda_{ij} \right) / \sum_{i \in X} p_i$$

where

λ^{e}_{XY} = Equivalent transition rate from subset X to subset Y

Network Reduction

- ANALYTICAL METHODS
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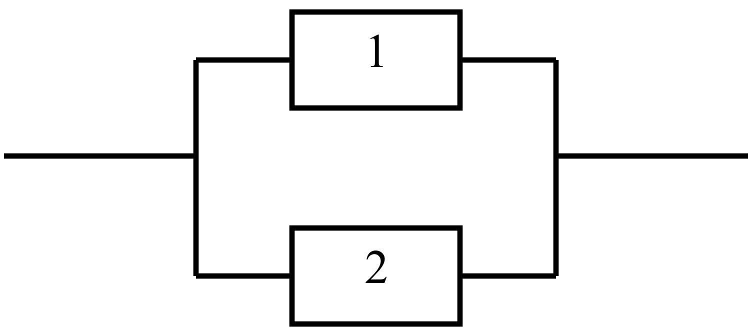
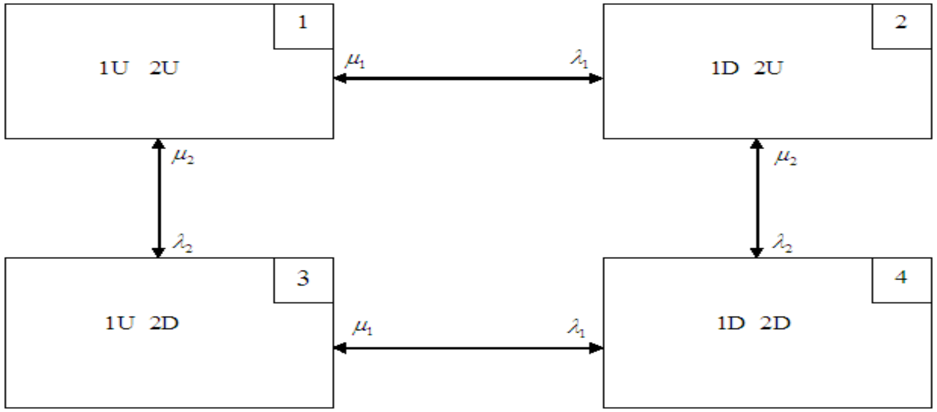
Parallel Systems

- Two components are said to be in parallel if both must fail for system failure.

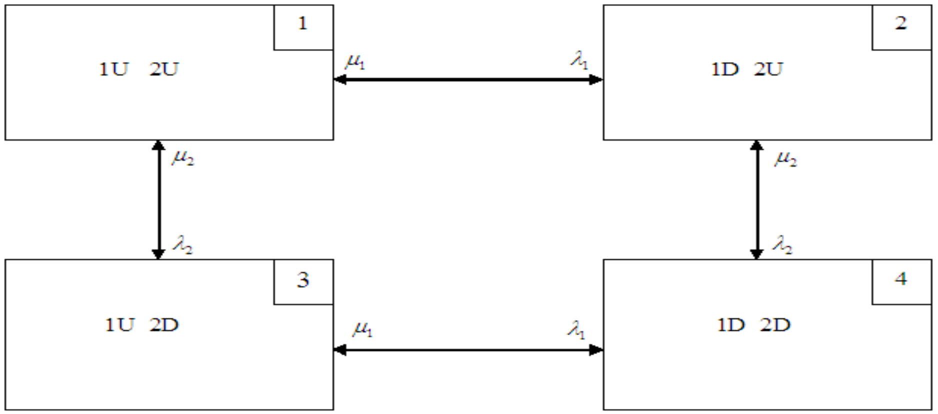




- Failure state: 4

Parallel Systems

- Failure state: 4



$$P_f = \frac{\lambda_1}{\lambda_1 + \mu_1} \times \frac{\lambda_2}{\lambda_2 + \mu_2}$$

$$f_f = \frac{\lambda_1}{\lambda_1 + \mu_1} \times \frac{\lambda_2}{\lambda_2 + \mu_2} (\mu_1 + \mu_2)$$

$$d_f = P_f / f_f$$

$$= \frac{1}{(\mu_1 + \mu_2)}$$

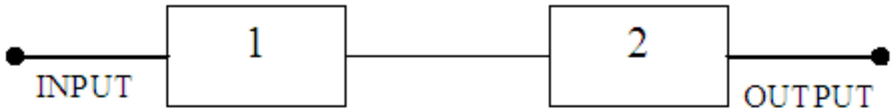
$$= \frac{r_1 r_2}{r_1 + r_2}$$

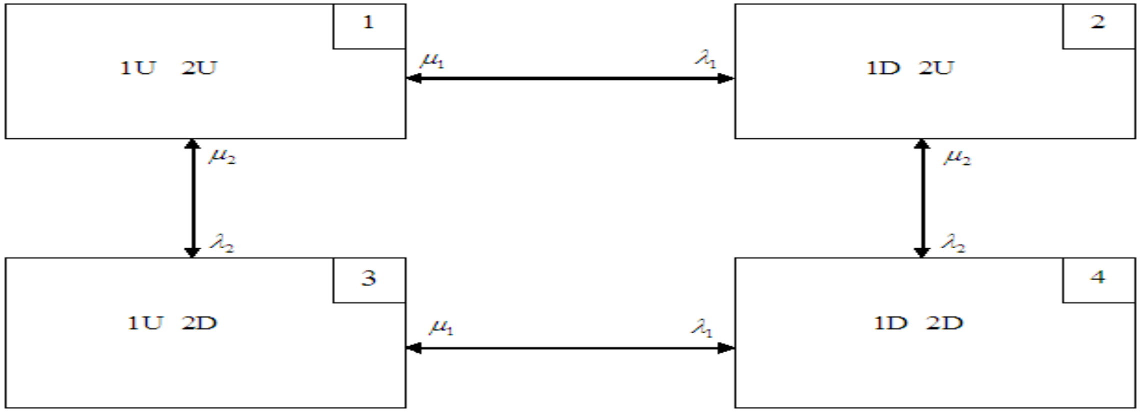
$$\lambda_f = f_f / (1 - P_f)$$

$$= \frac{\lambda_1 \lambda_2 (r_1 + r_2)}{1 + \lambda_1 r_1 + \lambda_2 r_2}$$

Series System

- Two components are said to be in series if the failure of either one causes system failure.

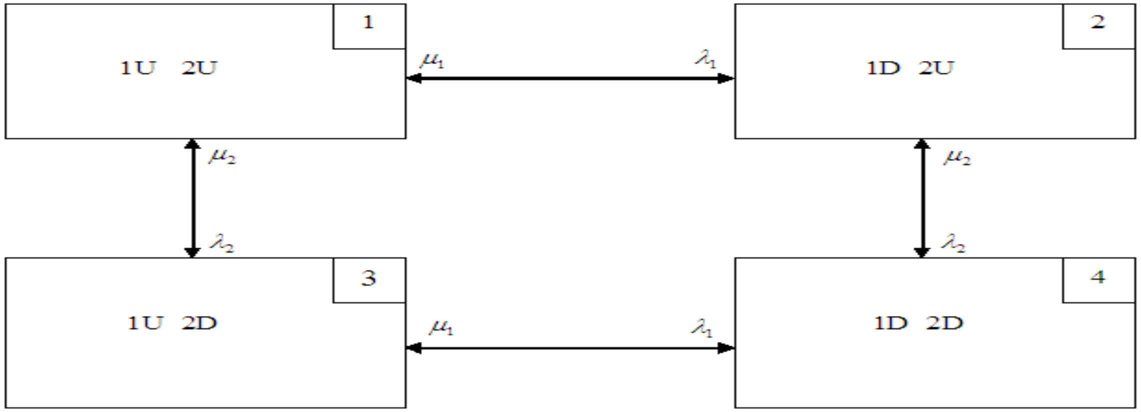




Series System

- Case 1: Independent Failure of Components

Failure state: 2,3,4



- Prob. of failure:

$$P_f = 1 - p_{1u}\ p_{2u}$$

p_{iu} = probability of component i being up.

- Frequency of Failure:

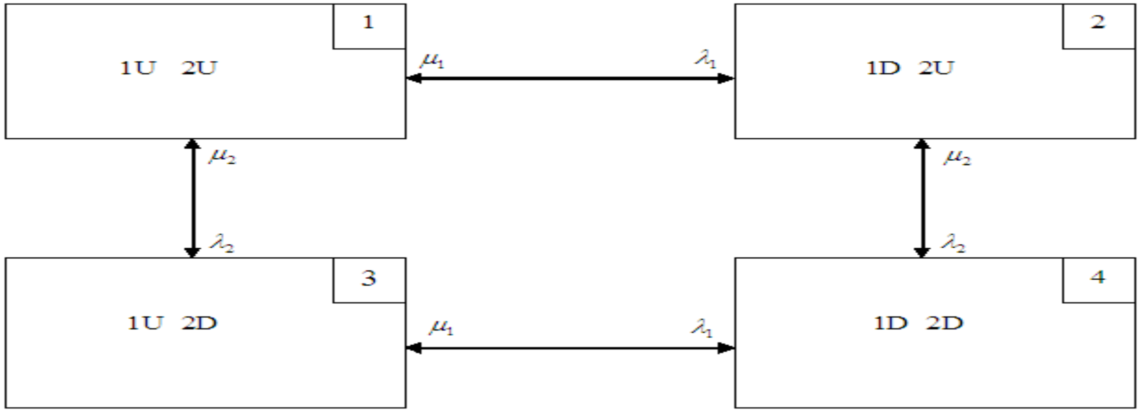
$$f_f = p_{1u}\ p_{2u}\ (\lambda_1 + \lambda_2)$$

$$= \frac{\mu_1}{\lambda_1 + \mu_1} \cdot \frac{\mu_2}{\lambda_2 + \mu_2} (\lambda_1 + \lambda_2)$$

Series System

- Case 1: Independent Failure of Components

Failure state: 2,3,4



- Failure rate:

$$\lambda_f = \text{failure rate}$$

$$= f_f / P_u$$

$$= \lambda_1 + \lambda_2$$

- Mean Down Time

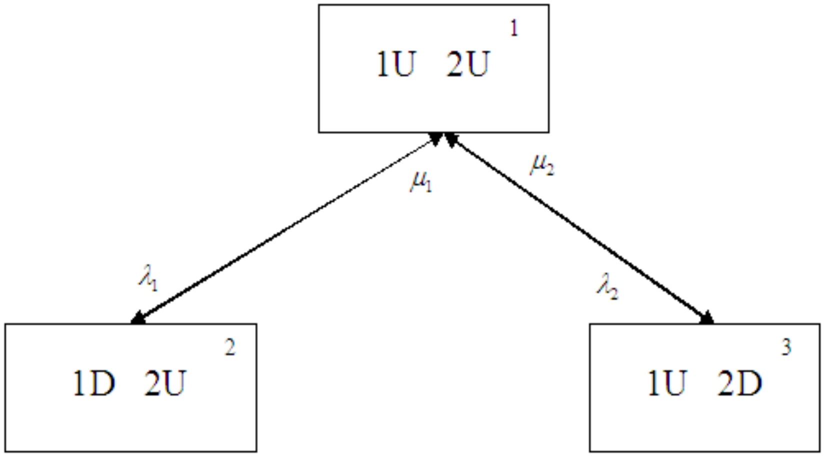
$$d_f = P_f / f_f = \frac{\lambda_1 r_1 + \lambda_2 r_2 + \lambda_1 \lambda_2 r_1 r_2}{\lambda_1 + \lambda_2}$$

r_i = mean down time of component i

Series System

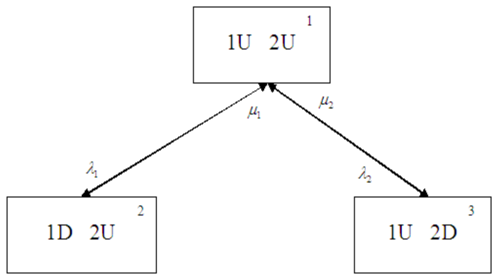
- Case 2: Dependent Failures
- Once system has failed, further component failures will not occur.

Failure state: 2,3



Series System

- Case 2: Dependent Failures
- Once system has failed, further component failures will not occur.



- Frequency balance equation
- For i=2,3

$$p_i \mu_{i-1} = p_1 \lambda_1$$

$$p_1 + p_2 + p_3 = 1$$

$$p_1 + p_1 \frac{\lambda_1}{\mu_1} + p_1 \frac{\lambda_1}{\mu_2} = 1$$

$$p_1 = \frac{1}{1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_1}{\mu_2}} = \frac{1}{Z}$$

Series System

- Case 2: Dependent Failures
- Once system has failed, further component failures will not occur.

- Prob. of failure

$$p_f = p_2 + p_3 = \frac{\frac{\lambda_1}{\mu_1} + \frac{\lambda_1}{\mu_2}}{Z}$$

- Freq. of failure

$$f_f = p_1(\lambda_1 + \lambda_2) = \frac{\lambda_1 + \lambda_2}{Z}$$

- Failure rate

$$\lambda_f = (\lambda_1 + \lambda_2)$$

- Mean down time

$$d_f = P_f / f_f$$

$$= \frac{\lambda_1 r_1 + \lambda_2 r_2}{\lambda_1 + \lambda_2}$$

- Prob. of failure

$$p_f = p_2 + p_3 = \frac{\frac{\lambda_1}{\mu_1} + \frac{\lambda_1}{\mu_2}}{Z}$$

- Freq. of failure

$$f_f = p_1(\lambda_1 + \lambda_2) = \frac{\lambda_1 + \lambda_2}{Z}$$

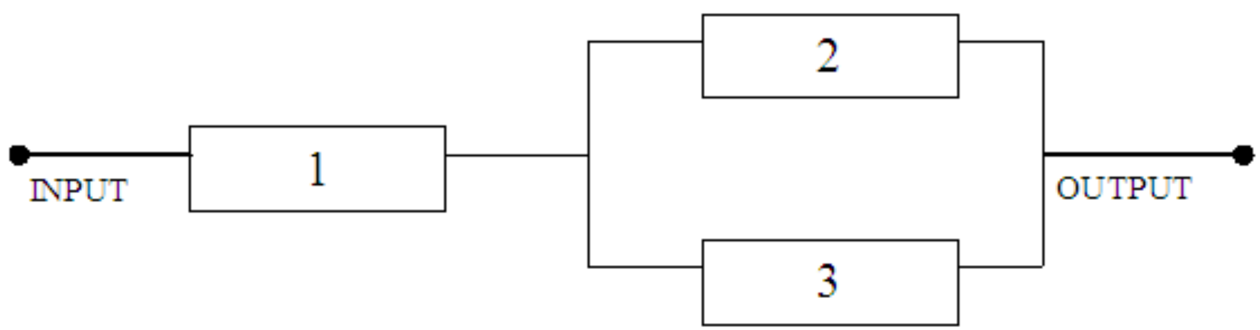
- Failure rate
- Mean down time

Network Reduction Method

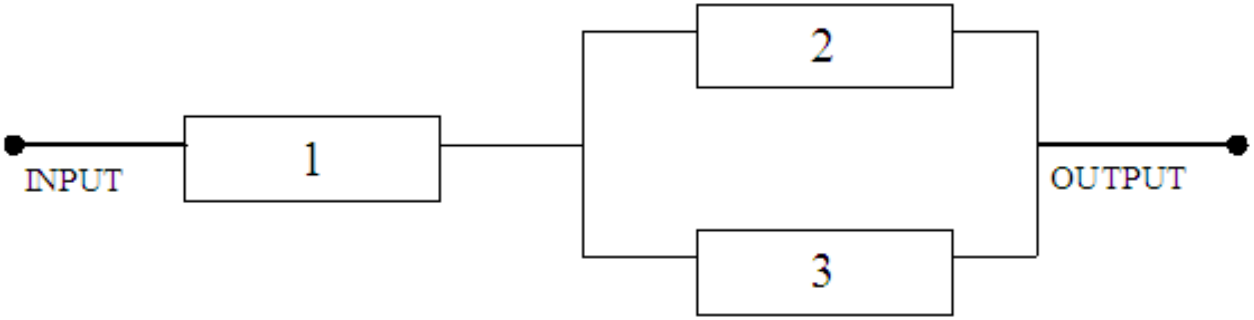
- Assume no subsystem is represented by more than one bloc.
- Replace all series blocks by an equivalent component.
- Replace all parallel blocks by an equivalent component.
- Repeat the above steps until the whole network reduces to an equivalent component.
- The system indices are those of the equivalent component.

Example Problem

- A system consists of one component in series with two components in parallel as shown. The failure rate of each component is 0.05 failure per year and the mean repair time is 10 hours.
- Using the network reduction method; calculate the failure rate and the mean down time of this system.



Example Problem



$$\lambda_{23} = (.05)(.05)\frac{(10+10)}{8760}\;\frac{1}{1+.05\times\frac{10}{8760}+0.05\times\frac{10}{8760}}$$

$$= .000005707 \text{ f/yr}$$

$$d = \frac{10\times 10}{20} = 5 \text{ hr.}$$

$$\lambda_{123} = .05 + .000005707$$

$$= .050005707$$

$$d_{123} = \frac{.000005707\times 5 + .05\times 10 + .000005707\times 0.05\times 10\times 5}{.05 + .000005707}$$

$$= 9.999 \text{ hr}$$

Practice Problems

- A system consists of three independent, identical components. At least two of the three components must work for system success. The mean up time of each component is 100 hours and the mean repair time is 10 hours. Determine:

(a) System unavailability.

(b) Frequency of system failure and the mean cycle time between successive failures.

(c) Mean up time and mean down time for the system.

Practice Problems

- A system consists of three generators, each 100 MW capacity. The failure and repair rates are 0.1 and .1 per day respectively. The load is 250 MW and constant. Assuming al the generators running at 0 hour, determine the probability of load loss at hour 24. The up and down times are exponentially distributed.

Min Cut Method

- ANALYTICAL METHODS
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Steps of Min Cut Method

- Definitions and Identification of cut sets

- Methods for calculation of indices:
  - Direct method
  - Block diagram method

Identification of Min Cut Set

- Min Cut set

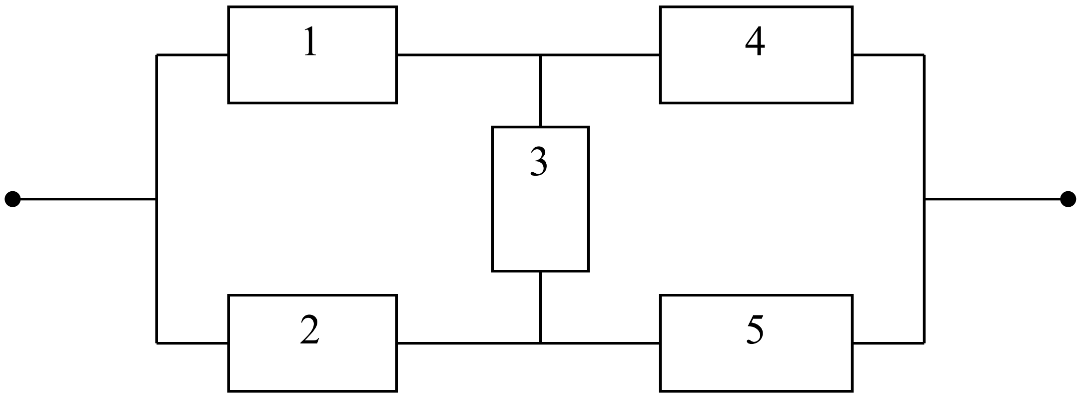
A cut set is set of components whose failure alone will cause system failure. A minimal cut has no proper subset of components whose failure alone will cause system failure.

- Min Tie or path Set

This is a set of components whose functioning alone will guarantee system success. A minimal tie set has no proper subset which is itself a tie set.

Identification of Min Cut Set

- Example System



Min Cut Set	Components in cut set
C_1	1, 2
C_2	4, 5
C_3	1, 3, 5
C_4	2, 3, 4

Identification of Min Cut Set

- Notes
  - In the definition of min cut set, a component means either a physical component or a condition.
  - The order of a cut set indicates number of components in the cut set. In the above example, C1 and C2 are second order cut sets and C3 and C4 are third order cut sets.
  - Generally cut sets above a certain order are ignored in the calculations as their probability may be negligible.

Direct Solution Method

- MIN CUT METHOD

- Methods for calculation of indices:
  - Direct method
  - Block diagram method

Expressions for Indices

Denote failure of all components in C_i by $\overline{C_i}$

$\cup$: Denotes "or"

$\cap$: Denotes "and"

Expressions for Indices

$$P_f = P(\overline{C_1} \cup \overline{C_2} \cup \overline{C_3} \ldots\ldots\ldots\ldots \cup \overline{C_m})$$

$$= P(\overline{C_1}) + P(\overline{C_2}) + P(\overline{C_3}) + \ldots\ldots\ldots\ldots\ldots\ldots + P(\overline{C_m}) \qquad \binom{m}{1} \text{ terms}$$

$$-[P(\overline{C_1} \cap \overline{C_2}) + P(\overline{C_1} \cap \overline{C_3}) + \ldots\ldots\ldots\ldots] \qquad \binom{m}{2} \text{ terms}$$

$$+[P(\overline{C_1} \cap \overline{C_2} \cap \overline{C_3}) + P(\overline{C_1} \cap \overline{C_2} \cap \overline{C_4}) + \ldots\ldots\ldots\ldots] \qquad \binom{m}{3} \text{ terms}$$

.

.

.

$$(-1)^{m-1}\, P(\overline{C_1} \cap \overline{C_2} \cap \overline{C_3} \cap \overline{C_4} \ldots\ldots\ldots\ldots \cap \overline{C_m}) \qquad \binom{m}{m} \text{ terms}$$

Expressions for Indices

$$f_f = \sum_{i=1}^{m} P(\overline{C_i})\overline{\mu_i}$$

$$- \sum_{i<j}^{m} P(\overline{C_i} \cap \overline{C_j})\overline{\mu}_{i+j}$$

$$+ \sum_{i<j<k}^{m} P(\overline{C_i} \cap \overline{C_j} \cap \overline{C_k})\overline{\mu}_{i+j+k}$$

.

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$$(-1)^{m-1} \quad P(\overline{C_1} \cap \overline{C_2} \cap \overline{C_3} \cap \overline{C_4} \ldots\ldots\ldots\ldots\ldots \cap \overline{C_m}) \quad \overline{\mu}_{i+j+\ldots\ldots m}$$

Where

$$\overline{\mu_i} = sum\ of\ \mu_i \text{ of all components} \in C_i$$

$$\overline{\mu}_{\text{i+j+......m}} = \sum_{l \in c_i \cup c_j \cup c_k \cup \ldots\ldots c_m} \mu_l$$

$$d_f = P_f / f_f$$

Explanation of terms by example system

- Consider two min cuts

$$C_1 = (1, 2)$$

$$C_3 = (1, 3, 5)$$

- Using upper bar to indicate failure

$$\overline{C_1} = \bar{1}, \bar{2} \qquad \overline{\mu}_1 = \mu_1 + \mu_2$$

$$\overline{C_3} = \bar{1}, \bar{3}, \bar{5} \qquad \overline{\mu}_3 = \mu_1 + \mu_3 + \mu_5$$

$$\overline{C_1} \cap \overline{C_3} = \bar{1}, \bar{2}, \bar{3}, \bar{5} \qquad \overline{\mu}_{1+3} = \mu_1 + \mu_2 + \mu_3 + \mu_5$$

Explanation of terms by example system

- Assuming independent component failures:

$$P(\overline{C_1}) = P(\bar{1})\ P(\bar{2})$$

$$P(\overline{C_3}) = P(\bar{1})\ P(\bar{3})\ P(\bar{5})$$

$$P(\overline{C_1} \cap \overline{C_3}) = P(\bar{1})\ P(\bar{2})\ P(\bar{3})\ P(\bar{5})$$

$$P(\overline{C_1}) \,.\, \overline{\mu_1} = P(\bar{1})\ P(\bar{2})\,(\mu_1 + \mu_2)$$

$$P(\overline{C_3}) \,.\, \overline{\mu_3} = P(\bar{1})\ P(\bar{3})\ P(\bar{5})\,(\mu_1 + \mu_3 + \mu_5)$$

$$P(\overline{C_1} \cap \overline{C_3})\ \overline{\mu}_{1+3} = P(\bar{1})\ P(\bar{2})\ P(\bar{3})\ P(\bar{5})\ (\mu_1 + \mu_2 + \mu_3 + \mu_5)$$

Explanation of terms by example system

- Bounds to indices

$$P_{U1} = \text{First upper bound to } P_f$$

$$= \sum_i P(\overline{C}_i)$$

$$P_{L1} = \text{First lower bound to } P_f$$

$$= P_{U1} - \sum_{i<j} P(\overline{C}_i \cap \overline{C}_j)$$

- By the successive addition of odd and even terms, increasingly closer upper and lower bounds are obtained.

$$f_{U1} = \text{First upper bound to } f_f$$

$$= \sum_i P(\overline{C}_i)\mu_i$$

$$f_{L1} = \text{First lower bound to } f_f$$

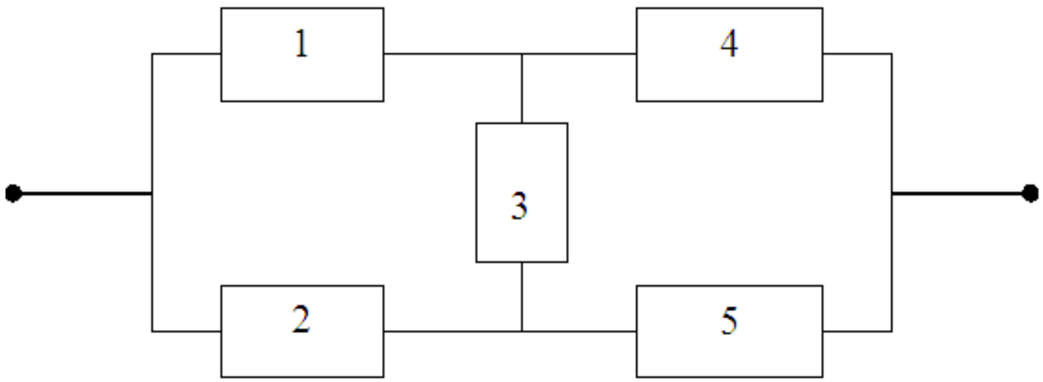
$$= f_{U1} - \sum_{i<j} P(\overline{C}_i \cap \overline{C}_j)\ \mu_{i+j}$$

Solution Using Block Diagrams

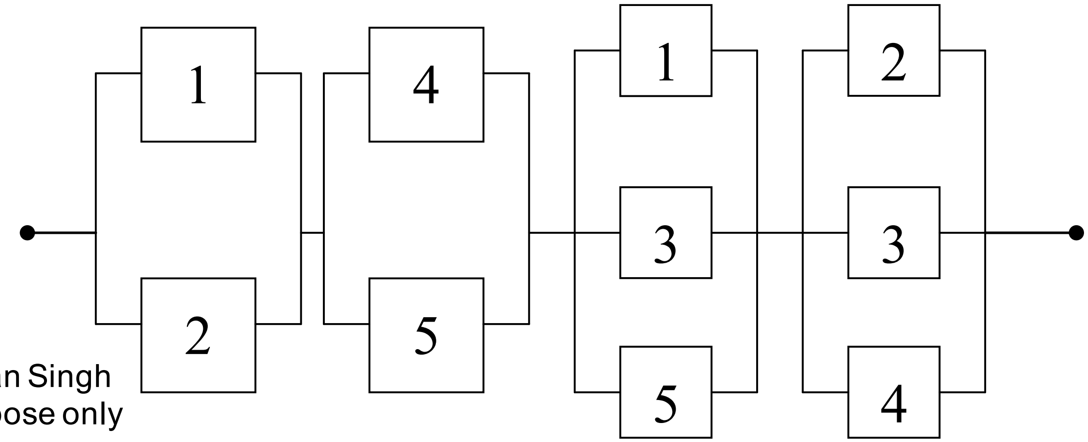
- MIN CUT METHOD

- Methods for calculation of indices:
  - Direct method
  - Block diagram method

Example



- Considering forced outages only and assuming “full” redundant capacity.
- Min-Cuts : {1,2}, {4,5}, {1,3,5}, {2,3,4}
- Visualization using Reliability Diagram:



Example

- Tabular procedure for quantitative evaluation:

Min Cut	1 Freq	2 Duration	(1×2) Prob.
1,2	$\lambda_1\lambda_2(r_1 + r_2)$	$\frac{r_1 r_2}{r_1 + r_2}$	$(\lambda_1 r_1)(\lambda_2 r_2)$
4,5	$\lambda_4\lambda_5(r_4 + r_5)$	$\frac{r_4 r_5}{r_4 + r_5}$	$(\lambda_4 r_4)(\lambda_5 r_5)$
1,3,5	$\lambda_1\lambda_3\lambda_5(r_1 r_3 + r_1 r_5 + r_3 r_5)$	$\frac{r_1 r_3 r_5}{r_1 r_3 + r_1 r_5 + r_3 r_5}$	$(\lambda_1 r_1)(\lambda_3 r_3)(\lambda_5 r_5)$
2,3,4	$\lambda_2\lambda_3\lambda_4(r_2 r_3 + r_2 r_4 + r_3 r_4)$	$\frac{r_2 r_3 r_4}{r_2 r_3 + r_2 r_4 + r_3 r_4}$	$(\lambda_2 r_2)(\lambda_3 r_3)(\lambda_4 r_4)$

$$f_{syst} = \sum f$$

$$D_{syst} = P_{syst} / f_{syst}$$

Summary of F & D expressions for min-cuts

- Overlapping Forced outages:

1st order:

$$f = \lambda_i$$

$$D = r_i$$

2nd order:

$$f = \lambda_i \lambda_j (r_i + r_j)$$

$$D = \frac{r_i r_j}{(r_i + r_j)}$$

3rd order:

$$f = \lambda_i \lambda_j \lambda_j (r_i r_j + r_i r_k + r_k r_j)$$

$$D = 1/(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k})$$

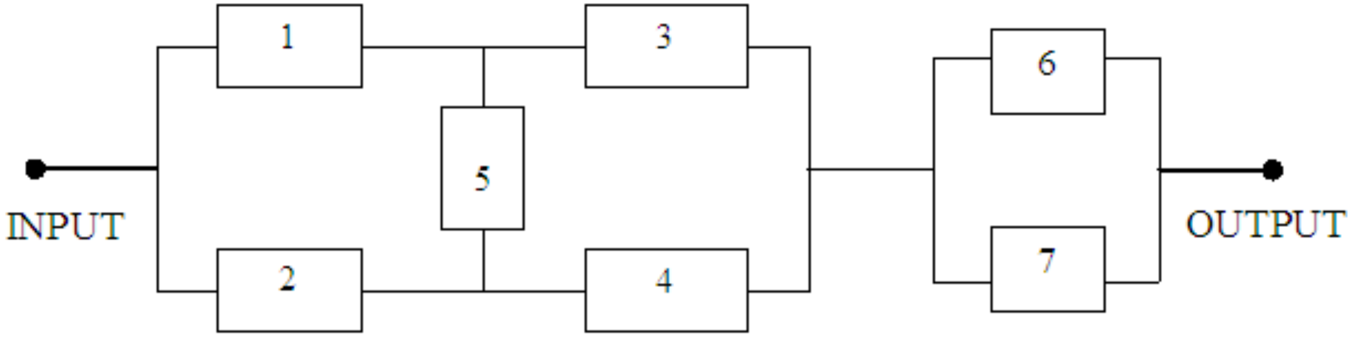
In general:

$$f = (\prod_i \lambda_i r_i) \sum_i 1/r_i$$

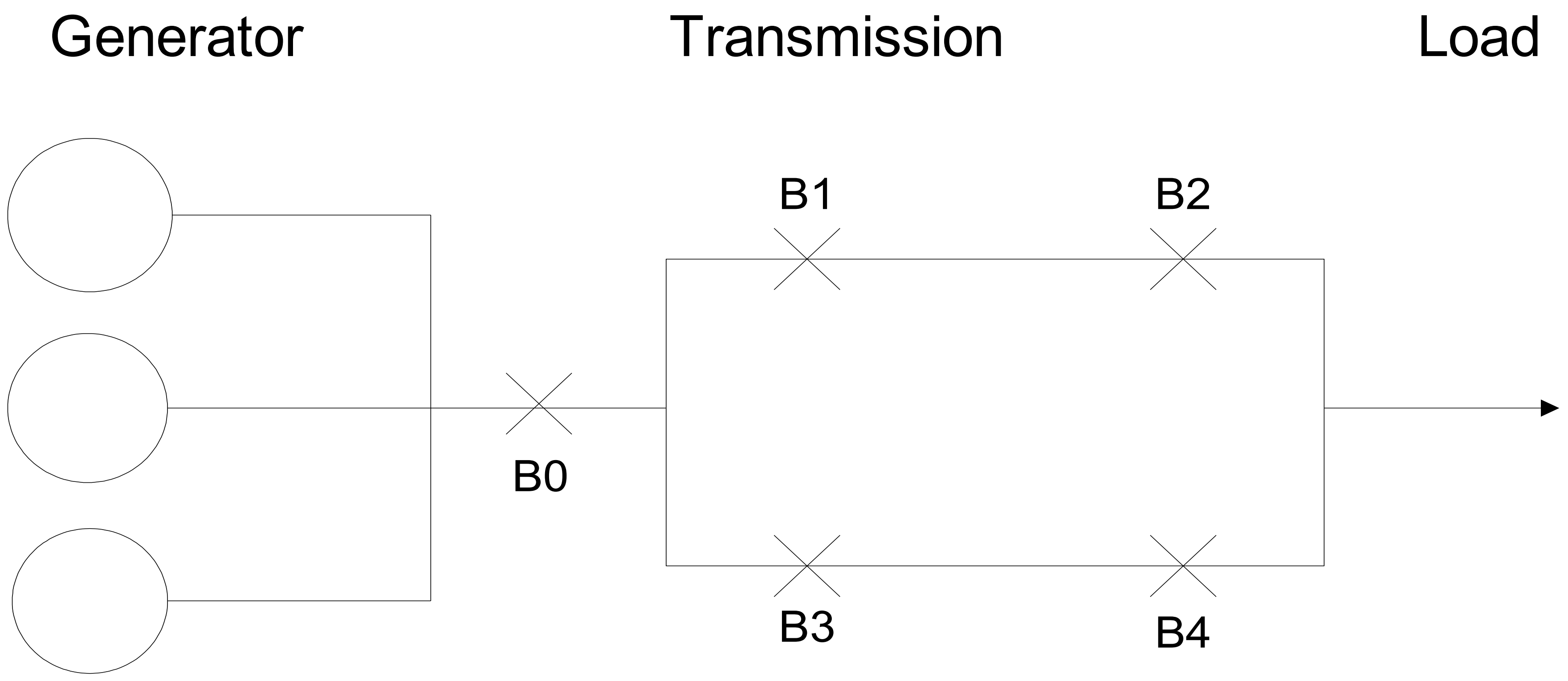
$$D = 1/\sum_i 1/r_i$$

Practice Problems

- For the network shown calculate the probability and frequency of failure.
- Assume all components independent and their failure and repair rates to be 0.1 and 0.5 per day.



EXAMPLE SYSTEM



EXAMPLE SYSTEM

- Generators:

Each generator either has full capacity of 50 MW or 0 MW when failed. Failure rate of each generator is 0.1/day and mean-repair-time is 12 hours

- Transmission Lines:

The failure rate of each transmission line is assumed to be 10 f/y during the normal weather and 100 f/y during the adverse weather. The mean down time is 8 hours. Capacity of each line is 100 MW.

- Weather:

The weather fluctuates between normal and adverse state with mean duration of normal state 200 hours and that of adverse state 6 hours.

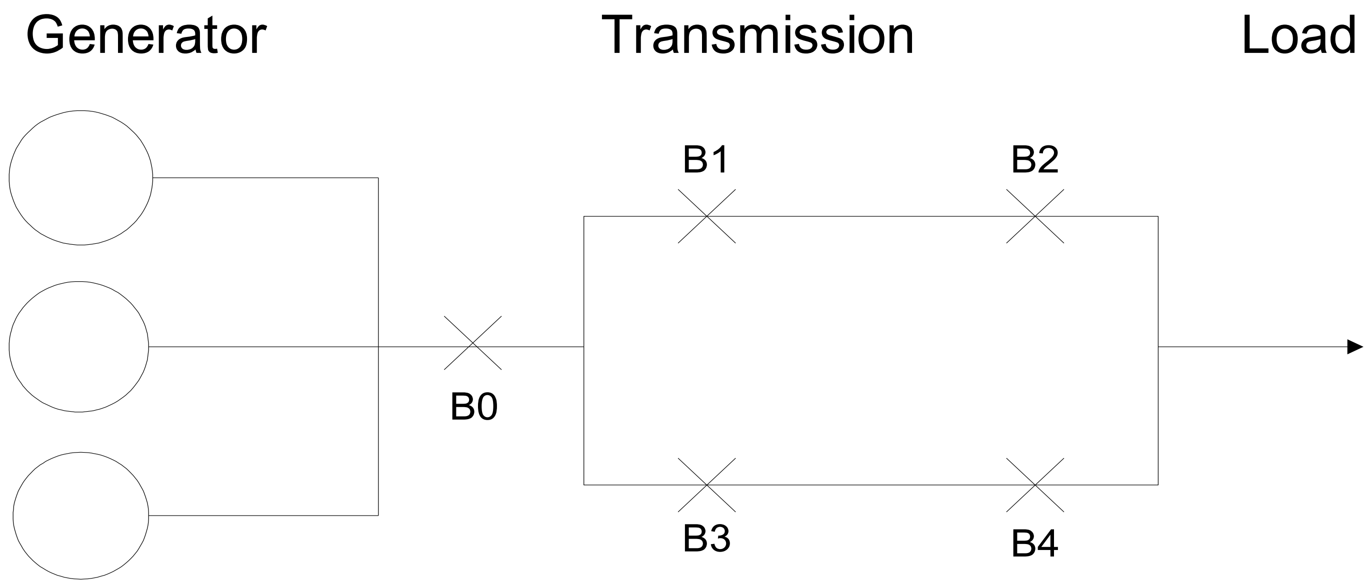
- Breakers:

Breakers are assumed perfectly reliable except that the pair B1&B2 or B3&B4 may not open on fault on the transmission line with probability 0.1.

- Load:

Load fluctuates between two states, 140 MW and 50 MW with mean duration in each state of 8hr and 16hr respectively.

EXAMPLE SYSTEM



- FOR THE DESCRIBED SYSTEM, HOW CAN YOU CALCULATE THE FOLLOWING BASIC RELIABILITY INDICES ?
  1. Loss of load probability
  2. Frequency of loss of load
  3. Mean duration of loss of load