LECTURE #3

WE CONTINUE INTERLUDE 1 :

- MISSION TIME
- HYBRID SYSTEMS

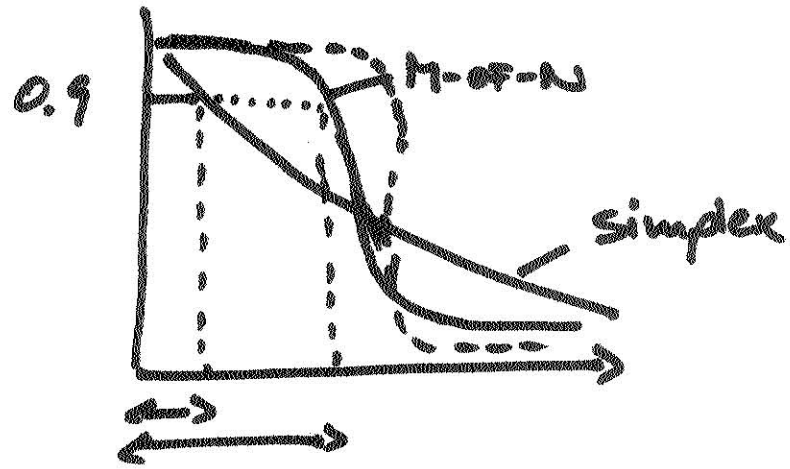
INTERLUDE #2 :

- MARKOV MODELING OF RELIABLE SYSTEMS.

<u>READING:</u> e.g. TRIVEDI
e.g. BEAUDRY

M-OUT-OF-N SYSTEMS (cont)

MISSION TIME:



MISSION TIME:

$$\boxed{\begin{aligned} R\left[MT(r)\right] &= r \\ MT\left(R(t)\right) &= t \end{aligned}}$$

Examples: (1) CONSTANT FAILURE RATE λ

$$\hookrightarrow \quad R(t) = e^{-\lambda t} = r$$

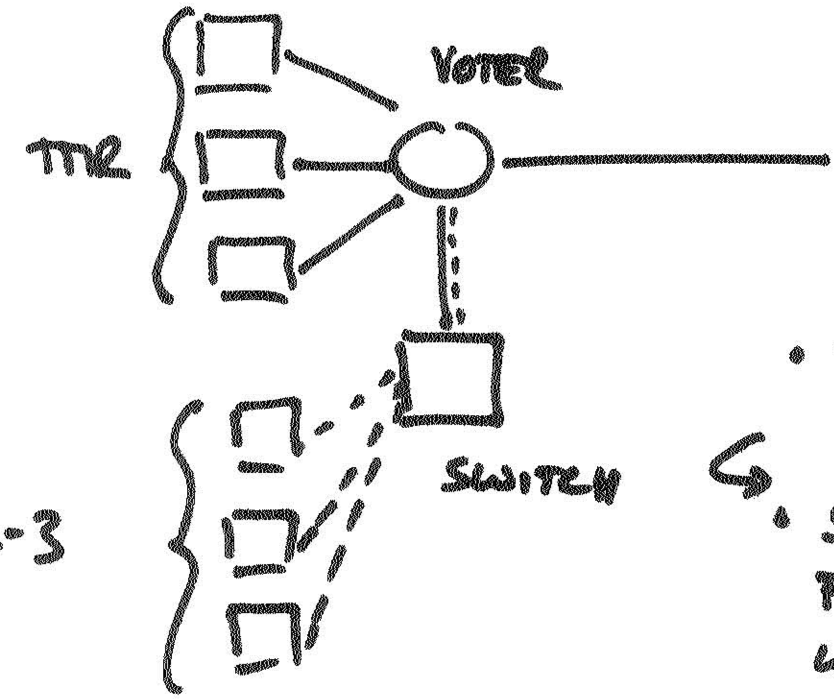
$$\hookrightarrow \quad MT(r) = \frac{-\ln r}{\lambda}$$

(2) NON-REDUNDANT SYSTEM WITH "n" COMPONENTS

$$MT(r) = \frac{-\ln r}{\sum \lambda_i}$$

HYBRID SYSTEMS:

EXAMPLE: TMR + n-3 SPARES



- VOTER DETECTS FAULTY MODULES
- SWITCH REPLACES FAULTY MODULE WITH SPARE

$$\Rightarrow \quad R_{HYBRID} = R_V \times R_{SW} \times \underbrace{\left\{ 1 - \binom{n}{1} R_M (1-R_M)^{n-1} \quad (1) \right.\\ \left. - (1-R_M)^n \right\} \quad (2)}_{P_r[\text{at most } n-2 \text{ FAULTS}]}$$

(1) P_r[n-1 FAULTS]

(2) P_r[n FAULTS]

WHAT ABOUT THE SWITCH?

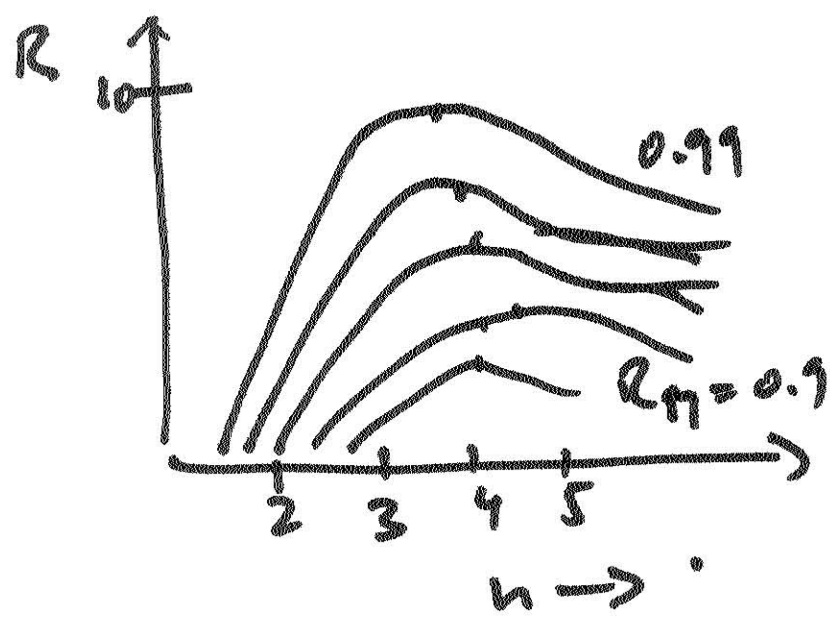
SWITCH COMPLEXITY INCREASES WITH "n" AS p^n

e.g. $R_{SW} = (R_M^{\alpha})^n$

$p = R_M^{\alpha} \quad \alpha = 0.1$

e.g. SWITCH COMPLEXITY = 10% OF MODULE COMPLEXITY.

$$\hookrightarrow R_{HYBRID} = R_M^{n\alpha} \left\{ 1 - n R_M (1-R_M)^{n-1} - (1-R_M)^n \right\}$$



MARKOV MODELING OF RELIABLE SYSTEMS

EXAMPLE:
- FOUR IDENTICAL MODULES + VOTER
- RELIABILITY OF EACH ELEMENT

$$R(t) = e^{-\lambda t}$$

- SYSTEM IS OPERATIONAL IF 2 ARE WORKING.

TWO METHODS TO DETERMINE R_{SYS} OR $R_{SYS}(t)$

1.) COMBINATORIAL METHOD "COUNTING GOOD STATES"

$$R_{SYS}(t) = \sum_{i=0}^{2} \binom{4}{i} [R(t)]^{4-i} [1 - R(t)]^{i}$$

2.) MARKOV METHOD

$$0 \xrightarrow{4\lambda} 1 \xrightarrow{3\lambda} 2 \xrightarrow{2\lambda} F$$

NO FAULTS | 1 FAULT | 2 FAULTS | SYSTEM FAILURE

- CONSIDER RATE OF TRANSITION
- DETERMINE PROB. OF EACH STATE.

MARKOV PROCESS MODELS

- ANALYZE COMPLEX PROBABILISTIC SYSTEMS
- STATE VS. STATE TRANSITIONS
- IN OUR CASE STATE := DISTINCT COMBINATION OF WORKING / FAILED MODULES.

  ↳ 2^n STATES FOR "n" MODULES.
- DISCRETE-TIME MODELS → TRANSITIONS OCCUR AT FIXED INTERVALS.
- CONTINUOUS-TIME MODELS → TRANSITIONS AT VARYING INTERVALS.
- FOR RELIABILITY MODELS, TRANSITION RATES ARE MODULE HAZARD FUNCTIONS AND REPAIR RATE FUNCTIONS MODIFIED BY COVERAGE FACTORS.

DISCRETE - PARAMETER MARKOV CHAIN

- MARKOV PROCESS IS KNOWN AS <u>MARKOV CHAIN</u> IF # STATES IS FINITE OR COUNTABLY INFINITE.

- SUPPOSE WE OBSERVE STATE OF SYSTEM AT A DISCRETE SET OF TIMES $t_1, t_2, \ldots, t_n$ WE OBSERVE RV'S $x_1, x_2, \ldots, x_n$

  ($x_i = j$ $\longrightarrow$ SYSTEM IN STATE j AT TIME t_i)

- MARKOV PROPERTY :

$$P_R[x_n = j_n \mid x_0 = j_0, x_1 = j_1, \ldots, x_{n-1} = j_{n-1}]$$

$$= P_R[x_n = j_n \mid x_{n-1} = j_{n-1}]$$

I.O.W. : PRESENT STATE DEFINES THE FUTURE (INDEPENDENT OF THE "PAST")

TRANS. PROB. MATRIX

$$P_j(n) := \Pr[X_n = j]$$

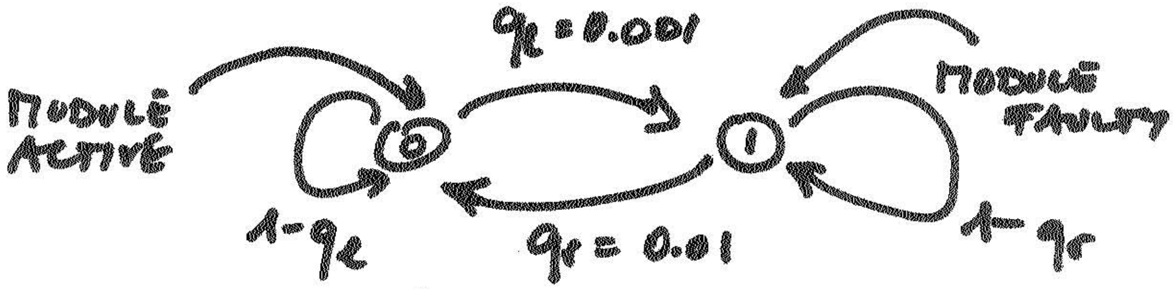
and

$$P_{jk}(m,n) := \Pr(X_n = k \mid X_m = j)$$

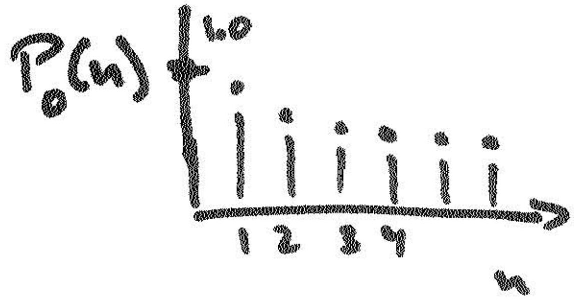
"HOMOGENEOUS" MARKOV CHAIN:

$P_{jk}(m,n)$ DEPENDS ON $m-n$,
NOT ON "m" OR "n".

EXAMPLE: TWO-STATE SYSTEM

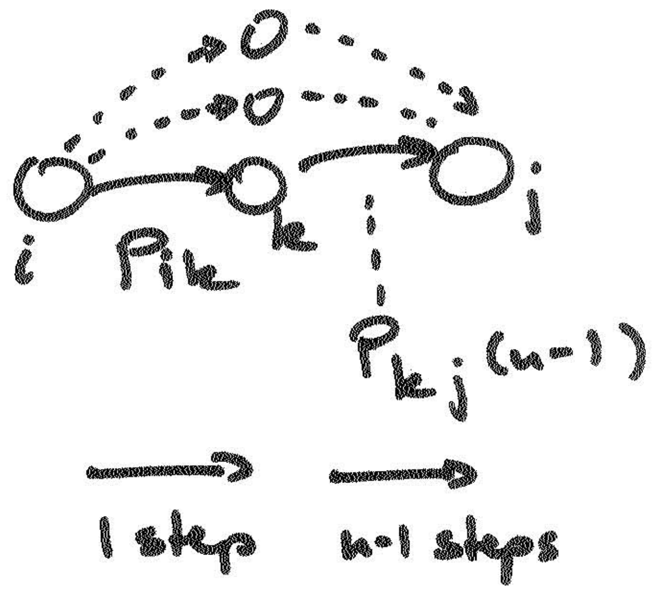


$$\underline{P} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-q_c & q_c \\ q_r & 1-q_r \end{bmatrix} \end{array}$$



COMPUTATION OF n-STEP TRANS. PROBS

$$P_{ij}(n) = \sum_k P_{ik}\, P_{kj}(n-1)$$



$$\hookrightarrow \quad p(n) = p(0)\, \mathbb{P}^n$$

$p(0)$: initial state vector

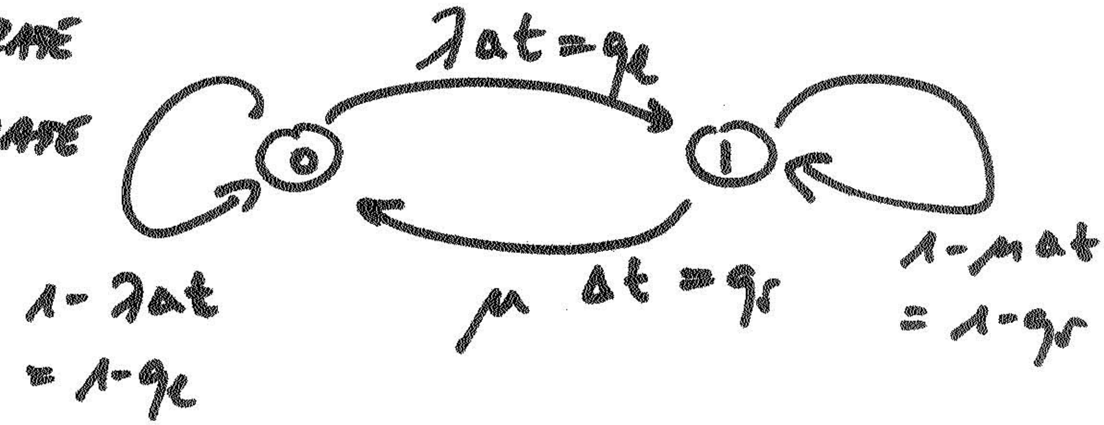
note: $P_0(n)$ = RELIABILITY

CONTINUOUS-TIME MARKOV CHAINS

DIFFERENTIAL MARKOV MODEL

λ = FAILURE RATE

μ = REPAIR RATE



$$P_{MATRIX} = \begin{bmatrix} 1-\lambda\Delta t & \lambda\Delta t \\ \mu\Delta t & 1-\mu\Delta t \end{bmatrix}$$

$$P_0(t+\Delta t) = (1-\lambda\Delta t) P_0(t) + \mu\Delta t \, P_1(t)$$

$$P_1(t+\Delta t) = \lambda\Delta t \, P_0(t) + (1-\mu\Delta t) P_1(t)$$

CONTINUOUS TIME MODEL:

$$\frac{dp_0}{dt} = \lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - p_0(t)}{\Delta t}$$

$$= -\lambda p_0(t) + \mu \, p_1(t)$$

$$\frac{dp_1}{dt} = \lim_{\Delta t \to 0} \frac{P_1(t+\Delta t) - p_1(t)}{\Delta t}$$

$$= \lambda p_0(t) - \mu \, p_1(t)$$

CONTINUOUS MODEL (CONT)

FOR EACH STATE i

$$\frac{dP_i}{dt} = -\underbrace{\sum_j r_{ij} P_i}_{\text{OUTGOING ARCS}} + \underbrace{\sum_k r_{ki} P_k}_{\text{INCOMING ARCS.}}$$

ANALYTIC SOLUTION
OF FIRST-ORDER DIFF EQ

GIVEN:

$$\begin{cases} \dfrac{dp_0}{dt} = -\lambda\, p_0(t) + \mu\, p_1(t) \\[2ex] \dfrac{dp_1}{dt} = \lambda\, p_0(t) - \mu\, p_1(t) \end{cases}$$

TAKE LAPLACE TRANSFORM $\quad L(f(t)) = \int_0^\infty f(t) e^{-st}\, dt$

$$\hookrightarrow \quad s p_0(s) - p_0(0) = -\lambda p_0(s) + \mu p_1(s)$$

$$s p_1(s) - p_1(0) = \lambda p_0(s) - \mu p_1(s)$$

$\uparrow$

initial values

$$[\,P_0(0) \quad P_1(0)\,] = [\,P_0(s) \quad P_1(s)\,] \begin{bmatrix} s+\lambda & -\lambda \\ -\mu & s+\mu \end{bmatrix}$$

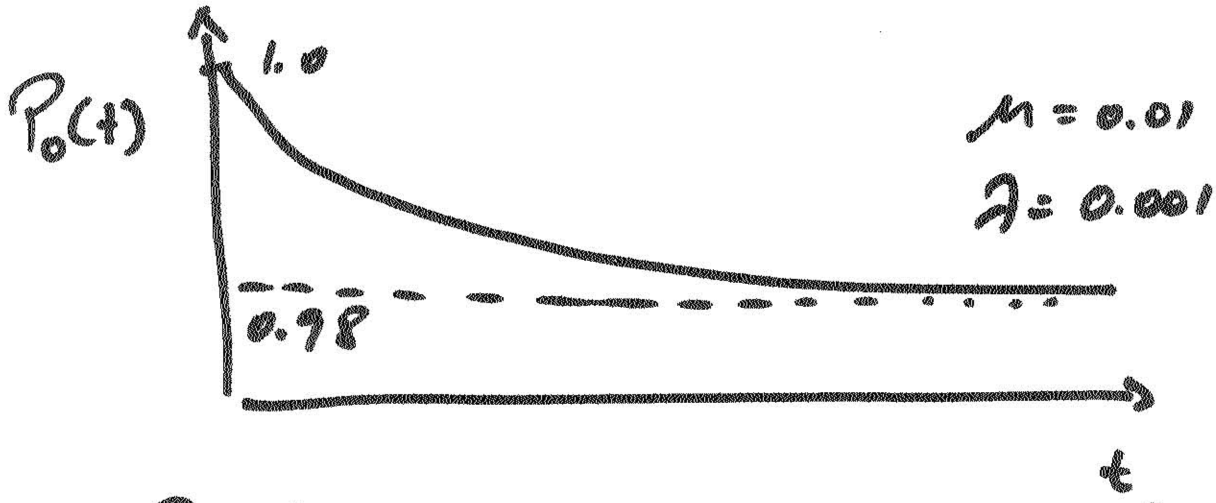
$$\hookrightarrow P_0(s) = \frac{s+\mu}{s^2+\lambda s+\mu s} \qquad P_1(s) = \frac{\lambda}{s^2+\lambda s+\mu s}$$

inverse transforms

$$\hookrightarrow P_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

(CONT)

$$P_1(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$



$P_0(t)$ GOES TO STEADY-STATE VALUE.

STEADY STATE CAN BE EASILY OBTAINED:

SETTING $\frac{dP_0}{dt}$ AND $\frac{dP_1}{dt} = 0$

$$\left\{ \begin{aligned} 0 &= -\lambda p_0 + \mu p_1 \\ 0 &= \lambda p_0 - \mu p_1 \end{aligned} \right\} \longrightarrow P_1 = \frac{\lambda}{\mu} P_0$$

$$p_0 + p_1 = 1$$

$$\hookrightarrow p_0 = \frac{\mu}{\lambda+\mu}$$