Useful Bounds

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Exponential Functions.

$$1 + x < e^x \quad \text{for all real } x \neq 0.$$

$$e^x(1 - x^2) \leq 1 + x \leq e^x \quad \text{for} \quad -1 \leq x \leq 1.$$

Logarithms.

$$\frac{-x}{1-x} \leq \ln(1-x) \leq -x \quad \text{for} \quad 0 < x < 1.$$

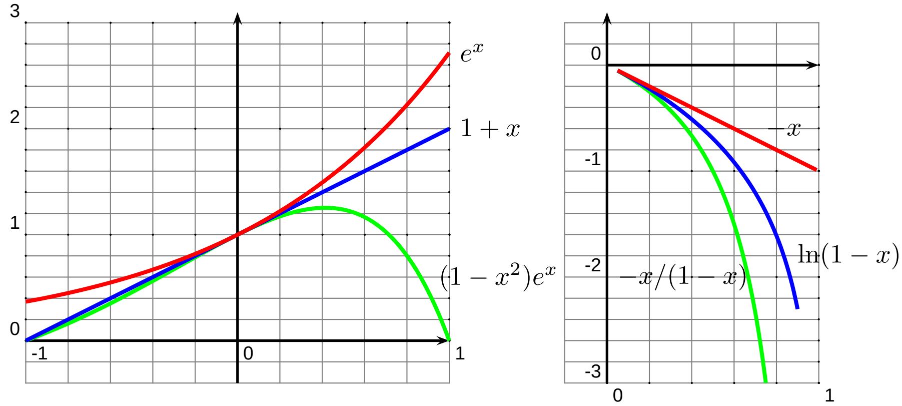

Figure 1: These function plots illustrate the bounds given above.

Factorials. Stirlings formula, easy version:

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq 2\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

A more advanced version of Stirlings formula is

$$n! = e^{\alpha_n}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

where $1/(12n+1) < \alpha_n < 1/12n$.

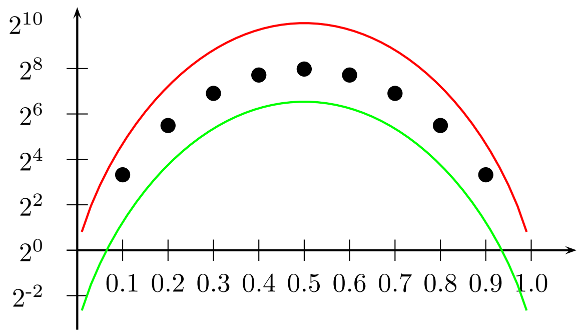

Figure 2: The graph shows the points at the coordinates $(k/10, \binom{10}{k})$ for all k in the range $0 < k < 10$, together with the lower and upper bounds (1) on the binomial coefficients.

Binomial Coefficients. A simple, but loose, bound:

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k.$$

A slightly more advanced bound is

$$\frac{2^{nH(k/n)}}{n+1} \leq \binom{n}{k} \leq 2^{nH(k/n)}, \tag{1}$$

where $H(q) = -q\log_2 q - (1-q)\log_2(1-q)$ is the binary entropy function.

As a consequence of Stirling's formula, we obtain for all k in the range $1 \leq k \leq n/2$ the formula

$$\binom{n}{k} = \beta(n,k)\left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k} \sqrt{\frac{n}{2\pi k(n-k)}},$$

where $e^{-1/6k} \leq \beta(n,k) \leq 1$.