Useful Notation

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When we discussed geometric random variables, we encountered the equality

$$\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\Pr[X=j]=\sum_{j=1}^{\infty}\sum_{i=1}^{j}\Pr[X=j], \tag{1}$$

which appears to be a bit puzzling. It turns out that with the proper notation, the manipulation of such sums becomes a routine matter.

Given a predicate *stmt*, we follow Iverson and use the bracket notation

$$[stmt]=\begin{cases}1 & \text{if } stmt \text{ is true,}\\ 0 & \text{otherwise.}\end{cases}$$

The bracket notation allows us to write summations without explicit bounds. As a simple example, let us convince ourselves that equation (1) is correct. We have

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\Pr[X=j] &= \sum_{i}\sum_{j}\Pr[X=j]\,[1\le i<\infty][i\le j\le\infty]\\ &= \sum_{j}\sum_{i}\Pr[X=j]\,[1\le i\le j<\infty]\\ &= \sum_{j=i}^{\infty}\sum_{i=1}^{j}\Pr[X=j].\end{aligned} \tag{2}$$

This notation makes such summation manipulations quite transparent. Let us have a look at another example. In the analysis of randomized quicksort, we encountered the equality

$$\sum_{i=1}^{n}\sum_{k=2}^{n-i+1}\frac{2}{k}=\sum_{k=2}^{n}\sum_{i=1}^{n+1-k}\frac{2}{k}. \tag{3}$$

Using the Iverson notation, one can explain this equality without hassle. Indeed,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{k=2}^{n-i+1}\frac{2}{k} &= \sum_{i}\sum_{k}\frac{2}{k}\,[1\le i\le n][2\le k\le n-i+1]\\ &= \sum_{i}\sum_{k}\frac{2}{k}\,[1\le i\le n-1][2\le k\le n-i+1] && i \text{ cannot attain } n\\ &= \sum_{i}\sum_{k}\frac{2}{k}\,[2\le n-i+1\le n][2\le k\le n-i+1] && \text{permute range of } i\\ &= \sum_{i}\sum_{k}\frac{2}{k}\,[2\le k\le n-i+1\le n] && \text{combine statements}\\ &= \sum_{k}\sum_{i}\frac{2}{k}[2\le k\le n][1\le i\le n+1-k] && \text{split into two parts}\\ &= \sum_{k=2}^{n}\sum_{i=1}^{n+1-k}\frac{2}{k}.\end{aligned}$$

For more details on the manipulation of sums, see [Graham, Knuth, Patashnik, *Concrete Mathematics*, Addison Wesley, 2nd edition, 1994].