A Unified View on Filter Banks

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ABSTRACT

An impressive variety of multirate filter banks evolved during the past twenty years. We present an algebraic approach that subsumes many concepts developed so far (e. g. multifilters, nonseparable multidimensional filter banks, cyclic filter banks, filter banks with values in finite fields, etc.). In our approach the signals and filters are viewed as elements of a group ring. We give necessary and sufficient conditions for perfect reconstruction and derive complete parametrizations in terms of ladder (or lifting) structures.

1. INTRODUCTION

Let us assume that we are given a monochrome image with a resolution of 512×512 pixels of 8 bit depth. The pixel values may be interpreted as real numbers, as integers, or even more specific as integers in the range [0..255]. In digital communication system applications it is not unusual to view the pixel values as elements of a finite field with 256 elements.

We want to process the image data with some sort of filter operation. For this reason we will assume throughout that the pixel values are given by elements of a ring A (such as the real numbers $\mathbf{R}$, the integers $\mathbf{Z}$, the residue class ring $\mathbf{Z}/256\mathbf{Z}$, or the finite field $\mathbf{F}_{2^8}$). We will always assume that this ring A is nonzero, associative, and contains an identity element 1. However, for reasons that will become apparent later, we do not restrict ourselves to commutative rings.

A row of our image may then be viewed as a sequence of ring elements or alternatively as an element of the Laurent polynomial ring $A[x, x^{-1}]$. Similarly, we may interpret a column of the image as an element of $A[y, y^{-1}]$, and hence the image may be regarded as an element of the ring

$$A[x, x^{-1}] \otimes_A A[y, y^{-1}] \cong A[x, x^{-1}, y, y^{-1}]. \tag{1}$$

However, this choice did not reflect the fact that our image has exactly 512 pixels per row (or column). We can emphasize this by interpreting the image data as an element of

$$A[x, x^{-1}]/\langle 1 - x^{512}\rangle \otimes_A A[y, y^{-1}]/\langle 1 - y^{512}\rangle \cong A[x, y]/\langle 1 - x^{512}, 1 - y^{512}\rangle. \tag{2}$$

Although the two image data types (1) and (2) seem to be markedly different, they share some important properties. Namely, they are both free A-modules generated by monomials $x^i y^j$ (a free A-module is the analogue of a vector space). Moreover, in both cases there is a group law on the exponents of these monomials: in the first case the exponents (i, j) of $x^i y^j$ are elements of the group $\mathbf{Z} \times \mathbf{Z}$ and in the second case the exponents (i, j) of $x^i y^j$ are elements of the group $\mathbf{Z}/512\mathbf{Z} \times \mathbf{Z}/512\mathbf{Z}$.

We can make this group structure even more apparent by introducing the well-known group ring concept. Given a ring A and a group G, then the group ring AG is defined to be the free left A-module with basis G. Thus the elements of this group ring are sums of the form

$$S = \sum_{g \in G} s_g\, g, \qquad s_g \in A,$$

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with at most finitely many $s_g \neq 0$. The A-module AG can be endowed with a multiplication operation by extending $(r_g g)(s_h h) = (r_g s_h)(gh)$ to all elements of AG by distributivity (with $r_g, s_g \in A$, $g, h \in G$). Thus the multiplication in G extends to a convolution in AG, as can be seen from the equation

$$\Big(\sum_{g\in G} s_g\, g\Big) \cdot \Big(\sum_{h\in G} t_h\, h\Big) = \sum_{h\in G} \Big(\sum_{g\in G} s_g t_{g^{-1}h}\Big) h.$$

The group ring concept allows us to re-interpret our two image data types (1) and (2). The Laurent polynomial ring (1) is isomorphic to the group ring $A[\mathbf{Z}\times\mathbf{Z}]$. Loosely speaking, the isomorphism $A[x, x^{-1}, y, y^{-1}] \to A[\mathbf{Z}\times\mathbf{Z}]$ simply maps the monomials $x^i y^j$ to their exponent vectors (i, j). The convolution operation in $A[\mathbf{Z}\times\mathbf{Z}]$ is given by the usual two-dimensional convolution

$$\Big(\sum_{(i,j)\in\mathbf{Z}\times\mathbf{Z}} s_{(i,j)}\,(i,j)\Big) \cdot \Big(\sum_{(n,m)\in\mathbf{Z}\times\mathbf{Z}} t_{(n,m)}\,(n,m)\Big) = \sum_{(n,m)\in\mathbf{Z}\times\mathbf{Z}} \Big(\sum_{(i,j)\in\mathbf{Z}\times\mathbf{Z}} s_{(i,j)} t_{(n-i,m-j)}\Big)(n,m).$$

The ring (2) is isomorphic to the group ring $A[\mathbf{Z}/512\mathbf{Z}\times\mathbf{Z}/512\mathbf{Z}]$. The convolution operation in this group ring is given by a two-dimensional "cyclic" convolution. Thus, the two image data types (1) and (2) basically differ in their convolution operations.

In the following, we describe a theory of multirate filter banks for signals and filters in a group ring AG. This will allow us to give a coherent treatment of seemingly different situations. Finite fields and rings are be of primary interest in digital communication, coding, or cryptography applications. Even in classical signal processing applications it is often desirable to deal with rings different from the real or complex numbers. For example, filters with integer coefficients are appealing for VLSI implementations with fixed point arithmetic. Even non-commutative rings are allowed, since we want to treat multifilter banks as well. Moreover, the group ring concept allows us to deal with higher-dimensional signals without too much hassle.

2. MULTIRATE OPERATIONS

Before we go further, we need to define the basic multirate operations: up- and downsampling. As we will see shortly, the downsampling operation is essentially a projection operation (although the representation is often changed by a relabeling isomorphism).

Let H be a normal subgroup of G of finite index $n := [G:H]$. The **τ-component** S_τ of a signal $S \in AG$ with respect to $H \subset G$ and $\tau \in G$ is defined by

$$S_\tau = \sum_{g\in H} s_{\tau g}\, g.$$

Let H' be a group isomorphic to H and denote by $r\colon AH \to AH'$ the induced ring isomorphism. Denote by e the neutral element of G. The **decimation operation** $[\downarrow]$ is defined by

$$[\downarrow]\colon \begin{cases} AG & \longrightarrow \quad AH', \\ S & \longmapsto \quad r(S_e). \end{cases}$$

Thus, the operation $[\downarrow]$ maps a signal $S \in AG$ onto its e-component S_e and changes (if necessary) the representation of the indices by use of the isomorphism $H \to H'$, $h \mapsto r(h)$. Clearly, the isomorphic change of representation from AH to AH' is irrelevant from an algebraic point of view. The **upsampling operation** $[\uparrow]$ is given by the isomorphism r^{-1}. Thus, decimation followed by upsampling gives the projection operation $S \mapsto S_e$.

The definitions of the operators $[\downarrow]$ and $[\uparrow]$ depend on the isomorphism $r\colon H \to H'$. In the following we will ignore this isomorphism for notational simplicity. This does not mean that we favour the choice $r\colon H \to H = H'$, $x \mapsto x$ in the definition of the decimation and upsampling operations. Other isomorphisms simply amount to a relabeling of the branches of the filter bank. The particular choice of the isomorphism r is not relevant here, since we are concerned with algebraic properties of filter banks.

3. FILTER BANKS

We introduce m-channel filter banks for signals in a group ring in this section. We will restrict ourselves to filter banks that use a uniform subsampling in each decomposition step.

Let A be a ring and G a group which contains a normal subgroup H of finite index $n = [G:H]$. A filter bank with m channels with respect to $AH \subset AG$ consists of m analysis filters $F^i \in AG$ and m synthesis filters $G^i \in AG$, $i \in [1..m]$. For easier reference, we will denote this data briefly by the tripel

$$\mathcal{F} := \left(AH \subset AG, (F^i)_{i\in[1..m]}, (G^i)_{i\in[1..m]}\right). \tag{3}$$

A filter bank with this data is sketched in Figure 1.

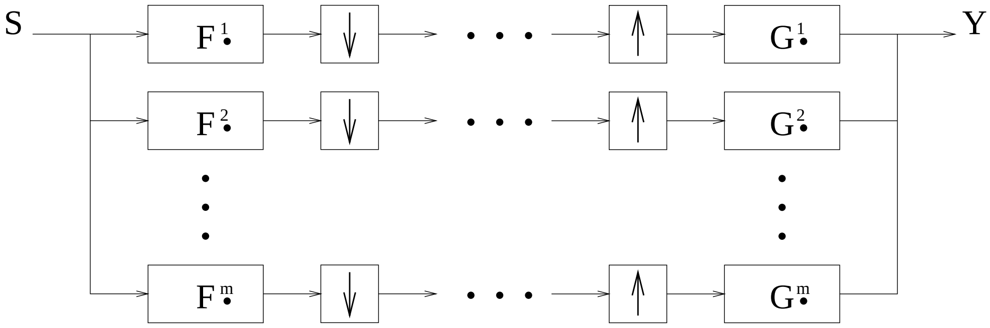

Figure 1. A filter bank for signals in the group ring AG with m channels.

An m-channel filter bank works as follows. The input signal S is multiplied with the analysis filters F^i (recall that multiplication in the group ring is a convolution). The resulting m intermediate signals $U^i := F^iS$ are then projected onto their e-components U^i_e by invoking a downsampling operation $[\downarrow]$ that is followed by a upsampling operation $[\uparrow]$. The sequences U^i_e are subsequently "interpolated" by convolution with the synthesis filters G^i. Finally, the interpolated sequences $G^iU^i_e$ are summed up, which yields

$$Y = \sum_{i=1}^{m} G^iU^i_e = \sum_{i=1}^{m} G^i(F^iS)_e. \tag{4}$$

4. PERFECT RECONSTRUCTION

A filter bank for signals in AG is called **perfect reconstructing** if and only if the output signal Y coincides with the input signal S for all $S \in AG$. We will study perfect reconstruction conditions in this section. As in classical filter bank theory, it is advantageous to introduce polyphase representations of signals and filters for this purpose.

Let H be a normal subgroup of G of finite index $n := [G:H]$. Denote by $\mathcal{T}$ a transversal of H in G, that is, the set $\mathcal{T}$ contains precisely one element from each coset of H in G. We will always assume that such a transversal contains the neutral element e of G. The **polyphase decomposition** of S with respect to $\mathcal{T}$ is then given by a sum of translated τ-components

$$S = \sum_{\tau\in\mathcal{T}} \tau S_\tau\,.$$

This is indeed an additive decomposition of the signal S, since we have

$$\sum_{\tau\in\mathcal{T}} \tau S_\tau = \sum_{\tau\in\mathcal{T}} \Big(\sum_{g\in H} s_{\tau g}\, \tau g\Big) = \sum_{\tau\in\mathcal{T}} \sum_{g\in \tau H} s_g\, g = \sum_{g\in G} s_g\, g.$$

The next theorem gives necessary and sufficient conditions for perfect reconstruction.

THEOREM 4.1. *Let $\mathcal{F}$ be an m-channel filter bank given by the data (3). Denote by $\mathcal{T}$ a transversal of H in G. The filter bank $\mathcal{F}$ is perfect reconstructing iff*

$$\delta_{\sigma,\tau} = \sum_{i=1}^{m} G^i_\sigma(\tau^{-1}F^i_{\tau^{-1}}\tau)$$

holds for all $\sigma, \tau \in \mathcal{T}$, where $\delta_{\sigma,\tau}$ denotes the Kronecker delta.

Proof. The idea of the proof is to replace the signals and filters in equation (4) by the corresponding polyphase decompositions; resorting terms and comparing coefficients will yield the result.

Let $\mathcal{T}$ be a transversal of H in G, then $\mathcal{T}^{-1} := \{\tau^{-1} \mid \tau \in \mathcal{T}\}$ is a transversal as well, since

$$(\tau H)^{-1} = H\tau^{-1} = (\tau^{-1}H\tau)\tau^{-1} = \tau^{-1}H$$

holds. We express the analysis filters $F^i = \sum_{\tau\in\mathcal{T}} \tau^{-1} F^i_{\tau^{-1}}$ with respect to the transversal $\mathcal{T}^{-1}$ and the signal $S = \sum_{\tau\in\mathcal{T}} \tau S^i_\tau$ with respect to the transversal $\mathcal{T}$.

Recall that the support of an element $S \in AG$ is defined by $\operatorname{supp} S = \{g \in G \mid s_g \neq 0\}$. Since the multiplication in the group ring is induced by the multiplication in the underlying group, we readily see that the support of the product $(\sigma F^i_\sigma)(\tau S_\tau)$ is contained in the coset $\sigma\tau H$. Consequently, the signal $U^i_e = [\uparrow][\downarrow](F^i S) \in AH$ can be written as

$$U^i_e = (F^i S)_e = \sum_{\tau\in\mathcal{T}} (\tau^{-1} F^i_{\tau^{-1}})(\tau S_\tau) = \sum_{\tau\in\mathcal{T}} \left(\tau^{-1} F^i_{\tau^{-1}} \tau\right) S_\tau.$$

Writing the synthesis filters G^i in polyphase form with respect to $\mathcal{T}$ yields now the following expression for Y:

$$Y = \sum_{i=1}^m G^i U^i_e = \sum_{i=1}^m \Big[\Big(\sum_{\sigma\in\mathcal{T}} \sigma G^i_\sigma \Big) \sum_{\tau\in\mathcal{T}} \left(\tau^{-1} F^i_{\tau^{-1}} \tau S_\tau\right) \Big].$$

Using distributivity and a change of sums, we obtain

$$Y = \sum_{\sigma\in\mathcal{T}} \sigma \Big(\sum_{\tau\in\mathcal{T}} \Big(\sum_{i=1}^m G^i_\sigma \tau^{-1} F^i_{\tau^{-1}} \tau \Big) S_\tau \Big).$$

Hence the polyphase components Y_σ with respect to $\mathcal{T}$ are given by

$$Y_\sigma = \sum_{\tau\in\mathcal{T}} \Big(\sum_{i=1}^m G^i_\sigma \tau^{-1} F^i_{\tau^{-1}} \tau \Big) S_\tau. \tag{5}$$

The signal S coincides with Y iff the polyphase components S_τ and Y_τ coincide for all $\tau \in \mathcal{T}$. Thus, perfect reconstruction of all signals $S \in AG$ is ensured if and only if

$$\sum_{i=1}^m G^i_\sigma (\tau^{-1} F^i_{\tau^{-1}} \tau) = \delta_{\sigma,\tau} \tag{6}$$

holds. □

5. POLYPHASE MATRICES

In this section we introduce the notion of polyphase matrices to give a convenient reformulation of Theorem 4.1. In contrast to classical filter bank theory, we also need a "twisted" form of polyphase matrices, since we allow nonabelian index groups.

Definition 5.1 (Polyphase Matrix). *Let $(F^i)_{i=1..m}$ be a sequence of filters of AG. Denote by $\mathcal{T} = (\tau_1, \dots, \tau_n)$ a transversal of H in G. Then the* ***polyphase matrix*** *of $(F^i)_{i=1..m}$ with respect to $\mathcal{T}$ is defined to be the $m\times n$-matrix*

$$(F^i_\tau)_{i,\tau} \qquad \textit{with} \qquad i = 1, \dots, m \ \textit{ and } \ \tau = \tau_1, \dots, \tau_n.$$

The ***twisted polyphase matrix*** *of $(F^i)_{i=1..m}$ with respect to the transversal $\mathcal{T}$ is defined to be the $m \times n$-matrix*

$$(\tau\, F^i_\tau\, \tau^{-1})_{i,\tau} \qquad \textit{with} \qquad i = 1, \dots, m \ \textit{ and } \ \tau = \tau_1, \dots, \tau_n.$$

In this paper we denote the twisted polyphase matrix of the analysis filters with respect to the transversal $\mathcal{T}^{-1}$ by H_{tp} and the polyphase matrix of the synthesis filters with respect to the transversal $\mathcal{T}$ by G_p. Using these conventions we can reformulate Theorem 4.1 as follows: the filter bank $\mathcal{F}$ is perfect reconstructing if and only if

$$G_p^t H_{tp} = I \tag{7}$$

holds. [$X^t \in \mathrm{Mat}_{n,m}(R)$ denotes the transposed matrix of $X \in \mathrm{Mat}_{m,n}(R)$; note that transposition does not map the coefficients to the opposite ring.] Hence the characterization of perfect reconstructing filter banks can be viewed as a problem of linear algebra over rings.

It is well-known that polyphase representations can be used to derive efficient implementations of filter banks. Equation (5) suggests to represent the input signal $S = \sum_{\tau\in\mathcal{T}} \tau S_\tau$ by its τ-component vector $(S_{\tau_1}, \ldots, S_{\tau_n})^t$. Left multiplication of this component vector with the matrix H_{tp} gives the result of the analysis filter bank. A subsequent multiplication with G_p^t yields the τ-component vector $(Y_{\tau_1}, \ldots, Y_{\tau_n})^t$ of the output signal Y. Figure 2 depicts the polyphase implementation of a filter bank following these calculation steps.

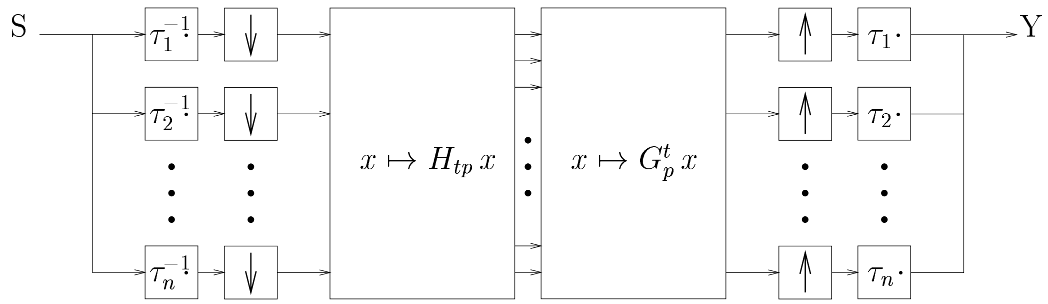

Figure 2. Polyphase implementation of a filter bank with data (3). Note that the projection on the τ-components of S with respect to $\mathcal{T}$ is realized by $S_{\tau_i} = \boxed{\downarrow}(\tau_i^{-1} S)$.

6. LADDER STRUCTURES

The preceding sections showed how perfect reconstruction of a filter bank can be ensured. The next step is to get a complete parametrization of all such filter banks, given the number of channels and the tower of group rings $AH \subset AG$. The corresponding mathematical problem is, loosely speaking, to express all pairs of polyphase matrices $(G_p^t, H_{tp},)$ satisfying (7) in terms of simple "basic building blocks". Popular choices for such building blocks are lattice structures[1–3] or ladder structures[4–6] (also dubbed lifting structures by Wim Sweldens). These structures are not only useful in the **design** of filter banks, but also in their **implementation**.

In the case of ladder structures one attempts to factor the twisted polyphase matrix $H_{tp} \in \mathrm{Mat}_{m,n}(AH)$ into a product of the following form:

$$H_{tp} = M_1 M_2 \cdots M_k \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \tag{8}$$

where the matrices $M_i \in \mathrm{GL}_m(AH)$ are elementary transvections or diagonal matrices. Recall that elementary transvections are defined as follows:

Definition 6.1. *Let R be a ring. A matrix in $\mathrm{GL}_m(R)$ is called* ***elementary transvection*** *if it coincides with the identity matrix except for a single off-diagonal entry. We denote by $B_{ij}(r) \in \mathrm{GL}_m(R)$ the elementary transvection with entry $r \in R$ at position (i, j), where $i \neq j$.*

Suppose that we are given the twisted polyphase matrix of the analysis filters in the product form (8), then the polyphase matrix G_p of the synthesis filters can be expressed in the following form:

$$G_p^t = (I_n\, 0) M_k^{-1} \cdots M_2^{-1} M_1^{-1}. \tag{9}$$

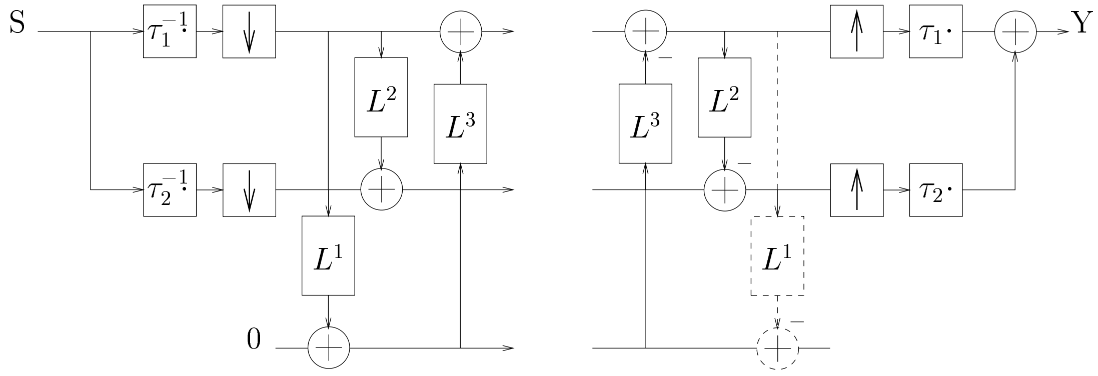

Figure 3. Ladder implementation of a three-channel filter bank for $\mathbf{R}[2\mathbf{Z}] \subset \mathbf{R}[\mathbf{Z}]$. The τ-components of the signal are the input of the upper two channels, a constant zero is the input of the lower channel. The synthesis filter bank is obtained by the same network of filters, applied in reverse order and with different sign. The rightmost (dashed) ladder step in the synthesis filter bank may be omitted, since the output of the lowest channel is ignored.

One feature of ladder structures is that the inverse matrices are of a particular simple form. Namely, if M_ℓ is given by $B_{ij}(r)$, with $r \in AH$, then $M_\ell^{-1} = B_{ij}(-r)$.

EXAMPLE 6.2. *Consider a three-channel filter bank for* $\mathbf{R}[2\mathbf{Z}] \subset \mathbf{R}[\mathbf{Z}]$. *Suppose the polyphase matrix* $H_{tp} \in \mathrm{Mat}_{3,2}(\mathbf{R}[2\mathbf{Z}])$ *is given by the following product*

$$H_{tp} = B_{13}(L^3)B_{21}(L^2)B_{31}(L^1)\begin{pmatrix} I_2 \\ 0 \end{pmatrix}.$$

The left-hand side of Figure 3 shows the analysis filter bank. Note that the ladder steps directly correspond to elementary transvections in the product above. The transposed synthesis polyphase matrix is given by the product

$$G_p^t = (I_2\, 0)B_{31}(-L^1)B_{21}(-L^2)B_{13}(-L^3),$$

which can be simplified to $G_p^t = (I_2\, 0)B_{21}(-L^2)B_{13}(-L^3)$.

DEFINITION 6.3. *Let R be a ring. We denote by $\mathrm{GE}_m(R)$ the subgroup of $\mathrm{GL}_m(R)$ generated by elementary transvections and invertible diagonal matrices. The ring R is called* $\mathbf{GE_m}$***-Ring*** *iff $\mathrm{GL}_m(R) = \mathrm{GE}_m(R)$ holds. If R is a GE_m-ring for all $m > 1$, then it is said to be a* ***generalized euclidean ring****.*

Given a pair $G_p^t, H_{tp} \in GL_m(AH)$ of polyphase matrices satisfying (7), it is natural to ask, if H_{tp} (and hence G_p^t) can be expressed as a product (8) of elementary transvections and invertible diagonal matrices. Unfortunately, the answer can be negative. In particular, if the group ring AH is not generalized euclidean, then we may construct counter examples.

7. CHARACTERIZATION OF POLYPHASE MATRICES

We have shown in §5 that a filter bank is perfect reconstructing if and only if the polyphase matrix G_p of the synthesis filters and the twisted polyphase matrix H_{tp} of the analysis filters satisfy the condition

$$G_p^t H_{tp} = I, \qquad \text{with} \qquad G_p^t \in \mathrm{Mat}_{n,m}(AH), H_{tp} \in \mathrm{Mat}_{m,n}(AH). \tag{10}$$

In this section we show how polyphase matrices satisfying (10) can be obtained from invertible matrices.

We need some notions from ring theory to formulate the following proposition. A ring R is said to have the **invariant basis property** iff $R^n \cong R^m$ implies $n = m$. This property is satisfied for any decent ring, e.g., for commutative rings or for noetherian rings. A module M is called **stably free** iff there exist finitely generated free modules F and F' such that $M \oplus F \cong F'$ holds. A ring R is called **hermite** iff all stably free R-modules are free.

Proposition 7.1. *Let AH be a non-trivial group ring satisfying the invariant basis property. Moreover, assume that all stably free AH-modules of rank $m-n$ with m generators are free. Let G_p and H_{tp} be matrices that satisfy the condition*

$$G_p^t H_{tp} = I \quad \textit{with} \quad G_p^t \in \mathrm{Mat}_{n,m}(AH),\ H_{tp} \in \mathrm{Mat}_{m,n}(AH).$$

Then there exists a matrix $X \in \mathrm{GL}_m(AH)$ such that

$$G_p^t = (I_n\, 0)\, X \quad \textit{and} \quad H_{tp} = X^{-1}\begin{pmatrix} I_n \\ 0 \end{pmatrix}$$

holds.

Thus, the matrix H_{tp} is given by the first n columns of X^{-1} and the matrix G_p^t is given by the first n rows of X.

Proof. Denote by α, β the homomorphisms that are induced by multiplying with the matrices H_{tp}, G_p^t from the left (with respect to the standard bases). We use the abbreviation $R := AH$. Then $\beta\colon R^m \to R^n$ is a split epimorphism and we obtain the split exact sequence

$$0 \longrightarrow \ker\beta \longrightarrow R^m \xrightarrow{\beta} R^n \longrightarrow 0.$$

Consequently, this sequence is isomorphic to a direct sum diagram[8] and we obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\beta \arrow[r] \arrow[d, equal] & R^m \arrow[r, shift right, "\beta"'] \arrow[d, "\mu"] & R^n \arrow[l, shift right, "\alpha"'] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker\beta \arrow[r, "\iota_1"] & \ker\beta \oplus R^n \arrow[r, shift left, "\pi_2"] & R^n \arrow[l, shift left, "\iota_2"] \arrow[r] & 0
\end{tikzcd}$$

where π_i denote the projection homomorphisms and ι_i the inclusion homomorphisms. The mapping $\mu\colon R^m \to R^n \oplus \ker\beta$ is a uniquely determined isomorphism. In particular, $\ker\beta$ is a stably free R-module of rank $m-n$ with m generators and thus free by assumption. Since R has the invariant basis property, it follows that $\ker\beta = R^{m-n}$ holds. From the diagram we see that $\alpha = \mu^{-1}\iota_2$ and $\beta = \pi_2\mu$ holds. The proposition just restates this fact in terms of matrices. □

We will see that many group rings of interest are hermite and have the invariant basis property. For hermite group rings the problem of parametrizing polyphase matrices reduces to the critically sampled case $m = n$, thanks to the previous proposition.

Remark 7.2. *Cohn[9] formulates a similar proposition under the stronger assumption that all projective AH-modules with m generators are free.*

8. SEMILOCAL GROUP RINGS

We will now give some examples of group rings $AH \subset AG$ that allow to implement all filter banks with the help of ladder structures. We assume in this section that H is given by a finite group. The main result of this section is given by the following theorem:

Theorem 8.1. *Let A be a semilocal ring and H a finite group. Then AH is a generalized euclidean ring. Therefore, any matrix in $\mathrm{GL}_n(AH)$ can be expressed as a product of elementary transvections and one diagonal matrix.*

We recall the relevant definitions. The Jacobson radical $J(R)$ of a ring R is defined to be the intersection of all maximal left ideals of R. The Jacobson radical is – in spite of the asymmetry in its definition – a two-sided ideal. A ring R is called **semilocal** iff $R/J(R)$ is a left artinian ring. It follows that $R/J(R)$ is a finite product of full matrix algebras over division rings. Finite rings, fields, artinian rings, and full matrix rings over semilocal rings are examples of semilocal rings.

Theorem 8.1 gives a complete parametrization of **all** polyphase matrices of perfect reconstructing filter banks over semilocal group rings, since all assumptions of Proposition 7.1 are satisfied:

Lemma 8.2. *A semilocal ring is hermite and has the invariant basis property.*

Proof. If a ring R lacks the invariant basis property and $\psi : R \to T$ is a ring homomorphism, then T lacks the invariant basis property as well, cf. [10]. Therefore, a semilocal ring R has the invariant basis property, since the homomorphic image $R/J(R)$ is left artinian and hence has the invariant basis property. Moreover, it follows from a proposition by Hyman Bass that all stably free modules over a semilocal ring are free, cf. [11] or [12,§20]. $\square$

The proof of Theorem 8.1 depends on the following key result:

LEMMA 8.3. *Let A be a semilocal ring and H a finite group. Then the group ring AH is again a semilocal ring.*

Proof. The Jacobson radical of A is contained in the Jacobson radical of AH, $J(A) \subset J(AH)$. This follows from a general fact about the behaviour of the Jacobson radical for inclusion homomorphisms, cf. Prop. 5.7 in.[12] Consequently, the two-sided ideal I in the group ring AH, which is generated by the set $J(A)$, satisfies the relation $I \subset J(AH)$.

We can interpret AH/I as a finitely generated $A/J(A)$-modul. Since $A/J(A)$ is left artinian, the module AH/I has to be left artinian too. In particular AH/I is a left artinian ring. It follows that the group ring AH is a semilocal ring. $\square$

The theorem follows from the preceding key lemma and the following known proposition, cf. [13,14]:

PROPOSITION 8.4. *A semilocal ring is a generalized euclidean ring.*

We omit the proof of this result here for the sake of brevity. More details will be given in [15].

REMARK 8.5. *As a special case we obtain a complete parametrization of perfect reconstructing cyclic filter banks with m channels. In our terminology these are filter banks for $AH \subset AG$ with $AH = \mathbf{C}[\mathbf{Z}/N\mathbf{Z}]$ and $AG = \mathbf{C}[\mathbf{Z}/mN\mathbf{Z}]$. Such filter banks have been investigated in [16,17]. Vaidyanathan and Kirac derive in [17] an (incomplete) parametrization of paraunitary cyclic filter banks.*

9. COMMUTATIVE GROUP RINGS

In this section we give more examples for the parametrization of perfect reconstructing filter banks. Now we consider commutative group rings AH, that is, the coefficient ring A is commutative and the group H is abelian.

Cyclic Groups. We start with the one-dimensional case, where H is given by the infinite cyclic group. In the case where the coefficient ring A is given by a field, the group ring AH is an euclidean ring. Daubechies and Sweldens show that the polyphase matrices can then be factored into elementary transvections and diagonal matrices.[18,19] We give a more general result in the following theorem:

THEOREM 9.1. *Let A be a commutative semilocal ring whose Jacobson radical $J(A)$ is nil, that is, all elements of $J(A)$ are nilpotent. Let H be the infinite cyclic group. Then AH is a generalized euclidean ring.*

Proof. The proof divides into two steps. In the first step we determine explicitly the Jacobson radical of AH. In the second step we decompose $AH/J(AH)$ in a product of simpler rings. We show that the GE_m-property of the ring AH can be derived from the GE_m-property of these "simpler rings".

According to Karpilovsky,[20] the Jacobson radical of AH is given by $J(AH) = N(A)H$, where $N(A)$ denotes the nil radical of A, that is, $N(A)$ contains all nilpotent elements of A. Therefore, we obtain $AH/J(AH) = (A/N(A))H$. By assumption $J(A) \subset N(A)$, hence we have $N(A) = J(A)$. This gives $AH/J(AH) = (A/J(A))H$.

The quotient ring $A/J(A)$ is semisimple, meaning that this ring can be decomposed as a direct sum of ideals $A/J(A) = A_1 \oplus \cdots \oplus A_n$, where each ideal A_i is a field. This yields the decomposition

$$(A/J(A))H \cong A_1H \times \cdots \times A_nH.$$

Each A_iH is a euclidean ring, hence a GE_m-ring for all $m > 0$. The direct product of GE_m-rings is a GE_m-ring.[14] Consequently, the ring $(A/J(A))H$ is a GE_m-ring for all $m > 0$. In general, a ring R is a GE_m-ring iff $R/J(R)$ is a GE_m-ring.[14] Therefore, we can conclude from the fact that $AH/J(AH)$ is generalized euclidean that AH is a generalized euclidean ring. $\square$

Now we want to check that Proposition 7.1 can be applied.

PROPOSITION 9.2. *Let A be a commutative semilocal ring with $J(A)$ nil and let H be the infinite cyclic group. The group ring AH is then hermite and has the invariant basis property.*

Proof. The group ring AH has the invariant basis property, since AH is commutative. We are left to prove that all stably free AH-modules are free. The fact that $AH/J(AH) = A/J(A)H$ holds is essential in the following proof.

Let us assume that $J(AH) = 0$. In this case AH is a finite direct product of euclidean ring. Thus, the group ring AH is in particular noetherian and has Krull dimension $\dim AH = 1$. Using a result by Bass, this implies that all stably free AH-moduls are free, cf. Theorem 7.3 in [21].

Now we consider the general case, i.e., $J(AH)$ may be arbitrary. We denote the reduction modulo $J(AH)$ by bars. Suppose we have a stably free AH-modul P that satisfies $P \oplus AH^r \cong AH^s$. It follows that

$$\overline{P} \oplus \overline{AH}^r \cong \overline{AH}^s$$

holds. Since $J(\overline{AH}) = 0$, all stably free $\overline{AH}$-moduls are free. Consequently, we obtain $\overline{P} \cong \overline{AH}^{s-r}$. Since we have reduced only modulo the Jacobson radical $J(AH)$, it follows that $P \cong AH^{r-s}$, cf. Prop. 6.6 in [22]. Therefore, all stably free AH-moduls are free. $\square$

EXAMPLE 9.3. *The previous theorem gives a complete parametrization of all perfect reconstructing filter banks with modular arithmetic $A = \mathbf{Z}/n\mathbf{Z}$ for one-dimensional signals. These filter banks find applications in lossless compression of image or medical volume data, cf. [23,24].*

Free Abelian Groups. Let A be a euclidean ring and F be a finitely generated free abelian group. The group ring AF is noetherian and thus has the invariant basis property. Moreover, due to a result by Suslin[25] we know that all stably free AF-modules are free. To summarize:

LEMMA 9.4 (SUSLIN). *Let A be a euclidean ring and F be a finitely generated free abelian group. Then the group ring AF is hermite and has the invariant basis property.*

Therefore, we can apply Proposition 7.1. From a more general result by Suslin[25] we derive a complete parametrization of filter banks over AH with at least three channels:

PROPOSITION 9.5 (SUSLIN). *Let A be a euclidean ring and let H be a free abelian group of finite rank. Then AH is a GE_m-Ring for all $m \geq 3$.*

Thereby we have obtained a complete parametrization of all perfect reconstructing filter banks for multidimensional signals, provided the filter bank has at least three channels. For two channels this parametrization is in general incomplete. Park[26] gives an algorithm that factors a polyphase matrix from $\mathrm{SL}_2(AH)$ into a product of elementary transvections, provided this is possible. He assumes that the coefficient ring A is a field. For filter banks with causal filters another method was developed by Tolhuizen, Hollmann, and Kalker.[27]

10. MULTIFILTER BANKS

The fast algorithms for multiwavelets are associated with multifilter banks, that is, filter banks with matrix-valued coefficients.[28] In multiwavelet applications the input is usually issued as a vector-valued sequence. We allow here matrix-valued input sequences (note that vector-valued sequences can be obtained as a special case by confining the input matrix coefficients to be zero except in the first column). We will show that multidimensional multifilter banks can parametrized in terms of lifting steps, complementing the results given in [29,30]. This result will be an easy consequence of the following theorem and proposition:

THEOREM 10.1. *Let A be a ring, and let H be a group. If the group ring AH is a GE_{nk}-ring, then the group ring $\mathrm{Mat}_k(A)H$ is a GE_n-ring. Thus, in particular, if the group ring AH is generalized euclidean, then so is the group ring $\mathrm{Mat}_k(A)H$ for any $k > 0$.*

Proof. We have to show that the group $\mathrm{GL}_n(\mathrm{Mat}_k(A)H)$ is generated by elementary transvections and invertible diagonal matrices, provided $\mathrm{GL}_{nk}(AH)$ is generated by such matrices. For this purpose it is convenient to look at the group ring $\mathrm{Mat}_k(A)H$ from a different perspective. More precisely, we make an isomorphic change from the group ring $\mathrm{Mat}_k(A)H$ to the matrix ring $\mathrm{Mat}_k(AH)$. [We will prove below that these rings are indeed isomorphic.]

The claim can be easily shown for the matrix ring $\mathrm{Mat}_k(AH)$. Indeed, we have the following group isomorphism

$$\mathrm{GL}_n(\mathrm{Mat}_k(AH)) \cong \mathrm{GL}_{nk}(AH).$$

By assumption the group on the right hand side is generated by elementary transvections and diagonal matrices, hence also the group on the left hand side. Thus $\mathrm{GL}_n(\mathrm{Mat}_k(A)H)$ is generated by transvections and diagonal matrices.

We are left to show that the group ring $\mathrm{Mat}_k(A)H$ is isomorphic to the matrix ring $\mathrm{Mat}_k(AH)$. Indeed, consider the mapping from the group H into the group of units of $\mathrm{Mat}_k(AH)$ given by

$$\chi\colon H \longrightarrow \mathrm{Mat}_k(AH), \quad g \longmapsto \mathrm{diag}(g, \dots, g)$$

and the inclusion homomorphism $\iota\colon \mathrm{Mat}_k(A) \longrightarrow \mathrm{Mat}_k(AH)$. These ring homomorphisms satisfy

$$\chi(g)\,\iota(r) = \iota(r)\,\chi(g), \qquad \text{for all} \qquad g \in H, r \in A.$$

By universality of the group ring $\mathrm{Mat}_k(A)H$, these two homomorphisms induce a unique ring homomorphism from $\mathrm{Mat}_k(A)H$ to $\mathrm{Mat}_k(AH)$, which is obviously bijective. $\square$

The next propostion shows that if the group ring AH has the invariant basis property and is hermite, then the group ring $\mathrm{Mat}_k(A)H$ has these properties as well.

PROPOSITION 10.2. *Let A be a ring, and let H be a group. If the group ring AH has the invariant basis property then $\mathrm{Mat}_k(A)H$ has the invariant basis property as well. The group ring $\mathrm{Mat}_k(A)H$ is hermite iff for all $n > 0$ every stably free AH-module of rank nk is free.*

Proof. It is again convenient to exploit the ring isomorphism $\mathrm{Mat}_k(A)H \cong \mathrm{Mat}_k(AH)$.

A ring R lacks the invariant basis property iff there exist matrices $S \in \mathrm{Mat}_{s,t}(R)$ and $T \in \mathrm{Mat}_{t,s}(R)$ with $s \neq t$ such that $ST = I_s$ and $TS = I_t$ holds.[10] Thus, if the ring $\mathrm{Mat}_k(A)H \cong \mathrm{Mat}_k(AH)$ lacks the invariant basis property then there exist matrices S and T, with $S \in \mathrm{Mat}_{ks,kt}(AH)$ and $T \in \mathrm{Mat}_{kt,ks}(AH)$, such that $ST = I_{ks}$ and $TS = I_{kt}$. Therefore, if $\mathrm{Mat}_k(A)H$ lacks the invariant basis property, then AH lacks the invariant basis property as well.

The matrix ring $\mathrm{Mat}_k(AH)$ is hermite iff for all $n > 0$ every stably free AH-module of rank nk is free. In fact, this statement is true for arbitrary rings R, cf. [31, p. 396]. Consequently, the group ring $\mathrm{Mat}_k(A)H \cong \mathrm{Mat}_k(AH)$ is hermite iff for all $n > 0$ every stably free AH-module of rank nk is free. $\square$

The previous two results allow us to translate the results on filter banks into the corresponding multifilter bank versions:

THEOREM 10.3. *Let A be a commutative semilocal ring whose Jacobson radical $J(A)$ is nil. Let H be the infinite cyclic group. Then the group ring $\mathrm{Mat}_k(A)H$ has the invariant basis property, is hermite, and generalized euclidean for all $k > 0$, $k \in \mathbf{Z}$.*

THEOREM 10.4. *Let A be a euclidean ring, let F be a free abelian group of finite rank. Then the group ring $\mathrm{Mat}_k(A)F$ has the invariant basis property, is hermite, and generalized euclidean for all $k > 1$, $k \in \mathbf{Z}$.*

11. CONCLUSION

We presented a unified filter bank theory that includes many types of filter banks developed so far. A group theoretic treatment of filter banks was also suggested by Kalker and Shah.[32] In our notation their approach is confined to signals from the group algebra $AG = \mathbf{C}F$, where F denotes a free abelian group of finite rank.

We emphasized the ring theoretic properties of the group ring in our approach. Many results concerning group rings and their stably free modules find direct applications in the structure theory of filter banks. In fact, the "filter bank theory" presented here may be viewed as a part of general linear algebra.

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