Twisted Filter Banks

Andreas Klappenecker

Texas A&M University, Department of Computer Science

College Station, TX 77843-3112, USA

klappi@cs.tamu.edu

Telephone: ++1 979 458 0608

September 7, 2004

Abstract

The main idea of a filter bank is to transform an input signal by subjecting it to some convolution operations, possibly followed by sampling rate reductions. We extend this idea to filter banks that are based on more general twisted convolution operations, which are not necessarily time invariant. Roughly speaking, a twisted convolution is obtained from the well-known convolution operations by allowing certain additional scalar factors in the multiplication operations. We discuss basic properties of these filter banks.

EDICS 1.TFSR

1 Introduction

The filter banks in this paper are based on so-called twisted convolutions, which generalize the usual convolution operations. We motivate this notion by a simple example that shows the essence of these operations. For simplicity, we confine ourselves to one-dimensional complex signals and filters (but we discuss this example with a view towards more general situations). We find it convenient to express the signals and filters as Laurent-polynomials with complex coefficients, that is, we adopt a z-transform point-of-view.

Suppose that the filter is given by a delay $f(z) = z^i$ and the signal $s(z) = z^j$ by a delta sequence. Recall that the usual convolution corresponds to $f(z)s(z) = z^{i+j}$. The result of a twisted convolution $f(z)\#s(z)$ for this signal and filter is given by

$$f(z)\#s(z) = \omega(i,j)z^{i+j}, \tag{1}$$

where $\omega(i,j)$ denotes a nonzero scalar factor. Without further constraints, this would be a rather unwieldy operation, since it is not necessarily associative. The associativity of # is guaranteed if the function $\omega\colon \mathbf{Z}\times\mathbf{Z}\to\mathbf{C}^\times$ satisfies

$$\omega(i,j)\omega(i+j,k) = \omega(j,k)\omega(i,j+k) \tag{2}$$

for all $i,j,k\in\mathbf{Z}$, as we will see in the next section. In addition, we will assume that ω satisfies the normalization condition

$$\omega(0,x) = \omega(x,0) = 1 \tag{3}$$

for all $x \in \mathbf{Z}$.

The twisted convolution # is in general a time-varying operation. However, the constraints (2) ensure that is does not behave completely arbitrary. And this allows us to develop a filter bank theory for twisted convolutions that closely resembles the traditional theory. There are some subtle changes, however, since # might fail to be commutative.

This paper is organized as follows. We will extend the twisted convolution operation to more general one-dimensional signals in the next section. We derive some basic properties of twisted convolutions in section 3. In section 4, we discuss two-channel filter banks based on twisted convolutions. In particular, we give necessary and sufficient conditions for perfect reconstruction of the input signals. We derive a simple implementation of a class of twisted convolutions in section 5. We show in section 6 that actually *all* twisted convolutions can be implemented with the method described in section 5; this is a peculiar property of the one-dimensional case.

2 Twisted Convolutions

The signals and filters in this section are Laurent polynomials with complex coefficients. We extend the twisted convolution (1) by linearity to signals and filters in $\mathbf{C}[z, 1/z]$. Thus, the twisted convolution of $f(z) = \sum_{i\in\mathbf{Z}} f_i z^i$ by $s(z) = \sum_{j\in\mathbf{Z}} s_j z^j$ is given by

$$f(z)\#s(z) = \sum_{i\in\mathbf{Z}}\sum_{j\in\mathbf{Z}} \omega(i,j) f_i\, s_j\, z^{i+j}, \tag{4}$$

which can also be written as

$$f(z)\#s(z) = \sum_{k\in\mathbf{Z}} \left[\sum_{j\in\mathbf{Z}} \omega(k-j,j) f_{k-j} s_j\right] z^k. \tag{5}$$

Lemma 1 *The twisted convolution # is an associative operation if and only if the condition (2) holds for all $i, j, k \in \mathbf{Z}$.*

Proof. Let $a(z) = z^i$, $b(z) = z^j$, and $c(z) = z^k$, then the equation $(a(z)\#b(z))\#c(z) = a(z)\#(b(z)\#c(z))$ implies that (2) must hold for $i, j, k \in \mathbf{Z}$. The converse follows directly from the definitions. Indeed, applying (4) to arbitrary Laurent polynomials $a(z) = \sum a_i z^i$, $b(z) = \sum b_j z^j$, $c(z) = \sum c_k z^k$ yields

$$(a(z)\#b(z))\#c(z) = \sum_{i,j,k} \omega(i,j)\omega(i+j,k) a_i b_j c_k z^{i+j+k}$$

and, using (2), we obtain

$$\sum_{i,j,k} \omega(j,k)\omega(i,j+k) a_i b_j c_k z^{i+j+k} = a(z)\#(b(z)\#c(z))$$

which proves associativity. □

From now on, we will assume that (2) is satisfied, so that # is always associative. We refer to the function $\omega\colon \mathbf{Z}\times\mathbf{Z}\to\mathbf{C}^\times$ associated with # as its factor set.

3 Twisted Calculus

We collect in this section a few simple rules that are helpful when dealing with twisted convolution operations. We focus only on elements that are useful in the following sections.

Twisted Delays. The operation # is associative, but it is not clear whether it is commutative. For instance, $a(z)\#z^\ell$ might fail to be the same as $z^\ell\#a(z)$. However, we can express $a(z)\#z^\ell$ in the form $z^\ell\#a'(z)$ as follows. Let $a(z)=\sum_{i\in\mathbf{Z}} a_i z^i$, then

$$\begin{aligned}\Big(\textstyle\sum_{i\in\mathbf{Z}} a_i z^i\Big)\#\, z^\ell &= \sum_{i\in\mathbf{Z}}\omega(i,\ell)a_i z^{i+\ell}\\ &= z^\ell\#\Big(\sum_{i\in\mathbf{Z}}\frac{\omega(i,\ell)}{\omega(\ell,i)}a_i z^i\Big).\end{aligned}$$

Notice that the coefficients of $a'(z)$ are readily obtained from the coefficients of $a(z)$. This calculation also shows that # is commutative if and only if $\omega(x,y)=\omega(y,x)$ holds for all $x,y\in\mathbf{Z}$.

A consequence of this fact is that $a^{(\ell)}(z) := z^{-\ell}\#a(z)\#z^\ell$ is given by

$$a^{(\ell)}(z)=\omega(-\ell,\ell)\sum_{i\in\mathbf{Z}}\frac{\omega(i,\ell)}{\omega(\ell,i)}a_i z^i$$

This will be useful in the next section on two-channel filter banks. The somewhat annoying scalar $\omega(-\ell,\ell)$ is a result of the simplification $z^{-\ell}\#z^\ell=\omega(-\ell,\ell)$.

Polyphase Decomposition. A polyphase decomposition of a signal is a very convenient tool in the analysis of multirate filter banks. For instance, we can write a Laurent polynomial in the form

$$a(z)=a_e(z^2)+z^\ell\#a_o(z^2) \tag{6}$$

where ℓ is an odd integer. Indeed, if $a(z) = \sum_{k\in\mathbf{Z}} a_k z^k$, then the even part is $a_e(z) = \sum a_{2k} z^k$ and the odd part is $a_o(z) = \sum_k \omega(\ell, 2k)^{-1} a_{2k+\ell} z^k$; the scalars $1/\omega(\ell, 2k)$ in the odd part compensate for the factors introduced by the operation $\#$.

Up- and Downsampling. Sampling rate conversions are done as usual. Reducing the sampling rate by a factor of two is done with the help of the operator $[2\!\downarrow]$. The effect of this operator on a Laurent polynomial $a(z)$ is

$$[2\!\downarrow]a(z) = a_e(z)$$

where $a_e(z)$ is the Laurent polynomial defined in (6). The effect of the upsampling operator $[2\!\uparrow]$ on a Laurent polynomial $a(z)$ is

$$[2\!\uparrow]a(z) = a(z^2).$$

Unfortunately, the sampling rate is essential in twisted convolutions. Although $[2\!\downarrow][2\!\uparrow]a(z) = a(z)$ holds, the result of

$$[2\!\downarrow]\Big(([2\!\uparrow]a(z)) \,\#\, ([2\!\uparrow]b(z^2)) \Big)$$

differs in general from $a(z)\#b(z)$. Thus, we cannot easily operate twisted convolutions of upsampled sequences at a lower sampling rate.

Twisted Laurent Polynomial Ring. Finally, we want to emphasize some properties of the data structure of twisted Laurent polynomials. We are particularly interested in the operations $+$ and $\#$. We note that these operations are connected by the distributive laws

$$a(z)\#(b(z) + c(z)) = a(z)\#b(z) + a(z)\#c(z),$$

$$(a(z) + b(z))\#c(z) = a(z)\#c(z) + b(z)\#c(z).$$

Let us denote by $\mathbf{C}^\omega[z, 1/z]$ the set of Laurent polynomials equipped with addition $+$ and with twisted convolution $\#$ as a multiplication rule. Then $\mathbf{C}^\omega[z, 1/z]$ is a ring. The multiplicative identity is 1, since $1\#x^\ell = \omega(0, \ell)x^\ell = x^\ell$ holds, as a result of the normalization condition (3).

4 Two-Channel Filter Banks

We describe in this section twisted two-channel filter banks with critical subsampling. Our main focus will be on the perfect reconstruction property.

Analysis. In the analysis part, an input signal $s(z)$ is subjected to twisted convolutions with two filters $f(z)$ and $g(z)$, followed by a sampling rate reduction. One obtains two intermediate signals $d_f(z) = [2\!\downarrow](f(z)\#s(z))$ and $d_g(z) = [2\!\downarrow](g(z)\#s(z))$.

Synthesis. In the synthesis stage, an upsampling operation is applied to the intermediate signals, twisted convolutions with synthesis filter are applied, and finally the two resulting signals are added. In other words, the resulting signal $r(z)$ is given by

$$r(z) = [\,\tilde{f}(z)\#[2\!\uparrow]d_f(z)\,] + [\,\tilde{g}(z)\#[2\!\uparrow]d_g(z)\,].$$

We will now focus on the following question: How do we have to choose the filters such that $r(z) = s(z)$ for all $s(z) \in \mathbf{C}^\omega[z, 1/z]$?

The polyphase decomposition will be helpful to answer this question. We find it convenient to write the input signal $s(z)$ in the form

$$s(z) = s_e(z^2) + z\#s_o(z^2),$$

and the analysis filter as

$$\begin{aligned} f(z) &= f_e(z^2) + z^{-1}\#f_o(z^2), \\ g(z) &= g_e(z^2) + z^{-1}\#g_o(z^2). \end{aligned} \tag{7}$$

Using these polyphase decompositions, the upsampled form of the intermediate signals $d_f(z)$ can be written as

$$\begin{aligned} A &= [2\!\uparrow][2\!\downarrow](f(z)\#s(z)) \\ &= f_e(z^2)\#s_e(z^2) + f_o^{(1)}(z^2)\#s_o(z^2), \end{aligned}$$

where $f_o^{(1)}(z^2) = z^{-1}\#f_o(z^2)\#z$, and the coefficients of this polynomial can be explicitly determined by the formula derived in subsection 3. Note that we used the fact that the terms

$[2{\downarrow}]z^{-1}\#f_o(z^2)\#s_e(z^2)$ and $[2{\downarrow}]f_e(z^2)\#z\#s_o(z^2)$ vanish. A similar calculation shows that the upsammpled result of the other channel is

$$\begin{aligned} B &= [2{\uparrow}][2{\downarrow}](g(z)\#s(z)) \\ &= g_e(z^2)\#s_e(z^2) + g_o^*(z^2)\#s_o(z^2). \end{aligned}$$

We also express the synthesis filters in terms of their polyphase components

$$\begin{aligned} \tilde{f}(z) &= \tilde{f}_e(z^2) + z\#\tilde{f}_o(z^2), \\ \tilde{g}(z) &= \tilde{g}_e(z^2) + z\#\tilde{g}_o(z^2). \end{aligned} \tag{8}$$

Applying the distributive law shows that $\tilde{f}(z)\#A$ is equal to

$$\begin{aligned} &\tilde{f}_e(z^2)\#\left[f_e(z^2)\#s_e(z^2) + f_o^{(1)}(z^2)\#s_o(z^2)\right] \\ &\quad +z\#\tilde{f}_o(z^2)\#\left[f_e(z^2)\#s_e(z^2) + f_o^{(1)}(z^2)\#s_o(z^2)\right] \end{aligned}$$

and similarly that $\tilde{g}(z)\#B$ is equal to

$$\begin{aligned} &\tilde{g}_e(z^2)\#\left[g_e(z^2)\#s_e(z^2) + g_o^{(1)}(z^2)\#s_o(z^2)\right] \\ &\quad +z\#\tilde{g}_o(z^2)\#\left[g_e(z^2)\#s_e(z^2) + g_o^{(1)}(z^2)\#s_o(z^2)\right]. \end{aligned}$$

Thus, we can write $r(z)$ in the form

$$(1, z)\begin{pmatrix} \tilde{f}_e(z^2) & \tilde{g}_e(z^2) \\ \tilde{f}_o(z^2) & \tilde{g}_o(z^2) \end{pmatrix}\begin{pmatrix} f_e(z^2) & f_o^{(1)}(z^2) \\ g_e(z^2) & g_o^{(1)}(z^2) \end{pmatrix}\begin{pmatrix} s_e(z^2) \\ s_o(z^2) \end{pmatrix}$$

where the multiplication of the terms is understood to be the twisted convolution $\#$, i.e., the arithmetic is over the twisted Laurent polynomial ring $\mathbf{C}^\omega[z, 1/z]$.

Summary. We need to introduce some terminology to record our result. Suppose that f and g are the analysis filters of the filter bank. The *twisted analysis polyphase matrix* is given by

$$H_p(z) = \begin{pmatrix} f_e(z^2) & f_o^{(1)}(z^2) \\ g_e(z^2) & g_o^{(1)}(z^2) \end{pmatrix},$$

where the entries are determined by (7). Recall that $a^{(1)}(z)$ denotes $z^{-1}\#a(z)\#z$, cf. Subsection 3. The *synthesis polyphase matrix* of the synthesis filters $\tilde{f}$ and $\tilde{g}$ is given by

$$G_p(z) = \begin{pmatrix} \tilde{f}_e(z^2) & \tilde{g}_e(z^2) \\ \tilde{f}_o(z^2) & \tilde{g}_o(z^2) \end{pmatrix},$$

where the entries are determined by (8). We can summarize our findings by the following theorem:

Theorem 2 *A critically subsampled twisted 2-channel filter bank is perfect reconstructing if and only if the synthesis polyphase matrix $G_p(z)$ is the left inverse of the twisted analysis polyphase matrix $H_p(z)$, that is, if and only if*

$$G_p(z)H_p(z) = \mathbf{1}$$

where $\mathbf{1}$ denotes the identity matrix in the matrix ring $\mathrm{Mat}_2(\mathbf{C}^\omega[z, 1/z])$.

5 Implementation with a Twist

We have not yet said anything about the implementation of a twisted convolution. The implementation of a convolution (without twist) is well-understood, so it seems natural to start from there. We will derive twisted versions with the help of a traditional convolution and some multipliers.

A convolution operation has two inputs and one output. We modify this convolution operation by pointwise multiplication of each input with a sequence $(\alpha(n))$ of nonzero complex coefficients, and pointwise multiplication of the output with the inverse sequence $(\alpha^{-1}(n))$. We only require that $\alpha(0) = 1$. Then, for instance, the input $a(z) = z^i$ and $b(z) = z^j$ yields the output

$$a(z)\#b(z) = \frac{\alpha(i)\alpha(j)}{\alpha(i+j)} z^{i+j}.$$

In other words, we obtained a twisted convolution with factor set

$$\omega(i,j) = \frac{\alpha(i)\alpha(j)}{\alpha(i+j)}. \tag{9}$$

One readily checks that the conditions (2) and (3) are indeed satisfied for our choice of ω.

6 Untwisting Weirdness

We gave an implementation for twisted convolutions of the form (9) in the previous section. In this section, we would like to get an idea how restrictive the assumption (9) is. The twisted convolutions in the previous section seem to be pretty specialized; for instance, if the factor set ω is of the form (9) then the associated twisted convolution operation $\#$ is commutative.

Notice that the method to derive a twisted convolution from an ordinary convolution can also be applied to twisted convolutions. We then obtain from a twisted convolution $\#_1$ with factor set ω_1 a new twisted convolution $\#_2$ with factor set ω_2 such that

$$\omega_2(i,j) = \frac{\alpha(i)\alpha(j)}{\alpha(i+j)}\omega_1(i,j). \tag{10}$$

We say that the twisted convolutions $\#_1$ and $\#_2$ are similar, $\#_1 \sim \#_2$, if and only if the exists a sequence of nonzero complex coefficients $(\alpha(n))_{n\in\mathbf{Z}}$, with $\alpha(0)=1$, such that the corresponding factor sets satisfy (10). It is easy to see that $\sim$ is an equivalence relation on the set of twisted convolutions.

Theorem 3 *All twisted convolution operations on one-dimensional complex Laurent polynomials are similar to the traditional convolution operation.*

Proof. Note that the factor sets, that is, the functions $\omega : \mathbf{Z}\times\mathbf{Z} \to \mathbf{C}^\times$ satisfying (2) and (3), form a group under pointwise multiplication; we denote this group by $Z^2(\mathbf{Z})$. The factor sets of the form (9) constitute a normal subgroup $B^2(\mathbf{Z})$ of the group $\mathbf{Z}^2(\mathbf{Z})$. The quotient group

$Z^2(\mathbf{Z})/B^2(\mathbf{Z})$ is known as the Schur multiplier of the cyclic group $\mathbf{Z}$, and it is isomorphic to the second cohomology group

$$H^2(\mathbf{Z}, \mathbf{C}^\times) \cong Z^2(\mathbf{Z})/B^2(\mathbf{Z}).$$

It is well-known that $H^2(\mathbf{Z}, \mathbf{C}^\times)$ is trivial, see [1, p. 58]. Thus, all factor sets are of the form (9). $\square$

7 Conclusions and Outlook

We have introduced a novel class of filter banks which are based on twisted convolutions instead of traditional convolutions. The same idea can be applied to higher dimensional filter banks [5], cyclic filter banks [4], filter banks over finite fields [3], and even filter banks over commutative rings [2]. Typically, we do not have an analogue of Theorem 3 in these cases. For instance, filter banks in higher dimensions admit many twisted convolution operations that are *not* similar to the traditional convolution. The derivation of the perfect reconstruction conditions, however, follow the same token as in the simple case presented in this paper. The twisted convolutions provide interesting examples of time-varying linear systems. We have outlined the basic theory of these filter banks. It is an interesting question whether the great flexibility of twisted filter banks can help to improve upon traditional filter bank applications.

References

[1] K.S. Brown. *Cohomology of Groups.* GTM 87. Springer-Verlag, New York, 1982.

[2] A. Klappenecker, M. Holschneider, and K. Flornes. Two-channel perfect reconstruction FIR filter banks over commutative rings. *Appl. Comp. Harm. Analysis*, 8:113–121, 2000.

[3] S.-M. Phoong and P.P. Vaidyanathan. Paraunitary filter banks over finite fields. *IEEE Trans. on Signal Processing*, 45:1443–1457, 1997.

[4] P.P. Vaidyanathan and A. Kirac. Cyclic LTI systems in digital signal processing. *IEEE Trans. on Signal Processing*, 47:433–447, 1999.

[5] M. Vetterli and J. Kovačević. *Wavelets and Subband Coding.* Prentice Hall, 1995.