Algebraic Wavelet Filters

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Abstract— **A method to compute the Discrete Wavelet Transform for certain wavelet filters is proposed that takes advantage of conjugacy properties in number fields. It is shown that wavelet filters derived from compactly supported orthonormal wavelets can be approximated with arbitrary precision by the proposed wavelet filters.**

Keywords—**Wavelets, Quadrature Mirror Filters (QMF), number fields.**

I. Introduction

In the last decade there has been a great interest in wavelet analysis. Discrete Wavelet Transforms (DWTs) found numerous applications in various different fields. The DWT for compactly supported orthonormal wavelets is attractive from an algorithmic point of view, since it can be implemented by Quadrature Mirror Filter (QMF) banks [1].

In this note a new technique for computing the DWT is introduced that exploits algebraic properties of special wavelet filters (see section III). It is shown that *all* wavelet filters derived from compactly supported orthonormal wavelets can be arbitrarily approximated by the proposed type of wavelet filters (see section IV). The method presented here often leads to a reduction of chip area in VLSI implementations of the DWT. This has been demonstrated for a specific example in [2].

II. Preliminaries

We consider square integrable compactly supported real-valued wavelets [3] in this note. Let (H, G) be a QMF pair

$$H(z) = \sum_{n=0}^{N-1} h_n\, z^{-n}, \qquad G(z) = \sum_{n=0}^{N-1} g_n\, z^{-n},$$

where $H(z)$ is a scaling filter and $G(z)$ is a wavelet filter. Recall that a QMF pair (H, G) derived from a multiresolution analysis has to meet the following scaling coefficient constraints [3]:

$$\sum_{n=0}^{N-1} h_{n-2k}\, h_n = \frac{1}{2}\,\delta_{k,0}\,, \quad k \in \mathbb{Z}\,, \tag{1}$$

where $\delta_{k,0}$ denotes the Kronecker delta,

$$\sum_{n=0}^{N-1} h_n = 1, \quad \text{and} \quad \sum_{n=0}^{N-1} g_n = 0. \tag{2}$$

The wavelet coefficients are related to the scaling coefficients by $g_n = (-1)^n\, h_{N-1-n}$.

In a basic decomposition step an input signal sequence (s_n) is convolved with the scaling and wavelet filters, followed by a decimation by two:

$$a_k = \sum_{n\in\mathbb{Z}} h_{n-2k}\, s_n, \qquad d_k = \sum_{n\in\mathbb{Z}} g_{n-2k}\, s_n.$$

III. Algebraic Wavelet Filters

In this section we present an alternative way to compute the DWT for wavelet filters that have algebraic filter coefficients. We start with a simple example to illustrate the underlying idea.

A. The Daubechies Wavelet D4

In signal processing applications the input signal samples are typically rational. We want to compute the DWT of such a sequence with respect to the famous Daubechies wavelet of order two, where the scaling filter coefficients are given by [3]:

$$h_0 = \frac{1+\sqrt{3}}{8}, h_1 = \frac{3+\sqrt{3}}{8}, h_2 = \frac{3-\sqrt{3}}{8}, h_3 = \frac{1-\sqrt{3}}{8}.$$

Assume now that we want to perform the calculations in an *exact* way, then we have to extend the field of rationals to a larger field that contains the quantity $\sqrt{3}$. Consequently, the *minimal necessary field* is given by the quadratic number field $\mathbb{Q}(\sqrt{3}) = \mathbb{Q}(h_0, h_1, h_2, h_3)$.

The wavelet filter $g_n = (-1)^n\, h_{N-1-n}$ is the "mirrored" version of the scaling filter h_n with every second sign changed. We will exploit this symmetry by use of conjugacy properties in $\mathbb{Q}(\sqrt{3})$.

The elements of the number field $\mathbb{Q}(\sqrt{3})$ can be written in the form $a + b\sqrt{3}$, where a and b are rationals. The non-trivial Galois automorphism σ of this number field maps an element $a + b\sqrt{3}$ to its conjugate $a - b\sqrt{3}$. Applying this automorphism to the scaling coefficient sequence (h_i) yields the "mirrored" sequence (h_{3-n}). Clearly, the "mirrored" scaling filter can be used in the computation of the wavelet filter. The details are given in the next subsection.

B. Algebraic Filter Structure

The wavelet filter $G(z)$ is derived from the scaling filter $H(z)$ by $G(z) = -z^{-k}\, H(-z^{-1})$, for some odd k. Alternatively, the wavelet filter may be obtained by applying the automorphism σ to the scaling filter with signed changed odd coefficients:

$$G(z) = \sigma H(-z).$$

Writing the scaling and wavelet filters in polyphase form will elucidate strong similarities in the structure of these filters. The scaling filter is given by

$$H(z) = H_{Even}(z^2) + z^{-1} H_{Odd}(z^2),$$

and the wavelet filter is almost the same:

$$G(z) = \sigma H(-z) = \sigma H_{Even}(z^2) - z^{-1} \sigma H_{Odd}(z^2).$$

We will choose a representation of the filter coefficients so that filters

$$H_{Even}(z) \quad \text{and} \quad H_{Odd}(z)$$

coincide with their conjugate filters

$$\sigma H_{Even}(z) \quad \text{and} \quad \sigma H_{Odd}(z)$$

respectively. This will allow us to re-use the same hardware for the computation of the scaling and the wavelet filter. We have to pay the price that the results of the scaling filter and the wavelet filter are represented in different ways.

The number field $\mathbb{Q}(\sqrt{3})$ can be interpreted as a two dimensional vector space over the rationals with basis $B = \{1, \sqrt{3}\}$. The scaling coefficients (h_i) with respect to this basis are:

$$m_0 = (\tfrac{1}{8}, \tfrac{1}{8}),\ m_1 = (\tfrac{3}{8}, \tfrac{1}{8}),\ m_2 = (\tfrac{3}{8}, -\tfrac{1}{8}),\ m_3 = (\tfrac{1}{8}, -\tfrac{1}{8}).$$

The mirrored sequence (h_{3-n}) has the *same* coefficients, if interpreted with respect to the conjugate basis $\sigma B = \{1, -\sqrt{3}\}$. Therefore, we define the vector-valued filter

$$M(z) = m_0 + m_1\, z^{-1} + m_2\, z^{-2} + m_3\, z^{-3}.$$

This work was supported by DFG under project Be 887/6-3.

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Applying this filter to a rational (scalar) input sequence yields a vector-valued output sequence. The projection of the output sequence onto its first vector-component yields the same result as the convolution of the input signal sequence with the filter

$$M_1(z) = 1/8 + 3/8\, z^{-1} + 3/8\, z^{-2} + 1/8\, z^{-3},$$

i.e., the projection of the filter $M(z)$ onto its first component. Similarly, the projection of the output sequence onto its second vector component yields the same result as the convolution of the input signal sequence with the filter

$$M_2(z) = 1/8 + 1/8\, z^{-1} - 1/8\, z^{-2} - 1/8\, z^{-3}.$$

Filtering a rational signal with $M_{Even}(z^2) + z^{-1}M_{Odd}(z^2)$ gives the scaling filtered signal, if we interpret the vector-valued output as numbers of the field $\mathbb{Q}(\sqrt{3})$ with respect to the basis B. Similarly, filtering a rational signal with $M_{Even}(z^2) - z^{-1}M_{Odd}(z^2)$ yields the wavelet filtered signal, if we interpret the vector-valued coefficients as elements of the number field $\mathbb{Q}(\sigma\sqrt{3}) = \mathbb{Q}(\sqrt{3})$ with respect to the basis σB.

As a result, we can implement the elementary decomposition step as shown in Fig. 1. The structure of the vector-valued filter (the dashed-box in Fig. 1) resembles Esteban-Galand like QMF filters [4]. The results a_k, d_k are converted back to a rational number representation in the box CVT. The details will be discussed in subsection III-D below.

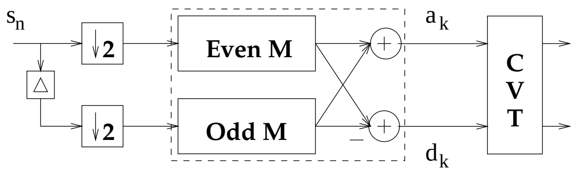

Fig. 1. Algebraic wavelet QMFs that satisfy the conjugacy property can be implemented with vector-valued filters.

C. General Situation

A wavelet filter is called *algebraic* iff all filter coefficients are algebraic, i. e., each filter coefficient is a zero of a polynomial with rational coefficients. Clearly, the wavelet filter is algebraic iff the scaling filter is algebraic.

Let (h_i) be a scaling filter. The field K obtained by adjoining the quantities h_i to the rationals is called the *minimal necessary field* for the filter (h_i), i. e.,

$$K = \mathbb{Q}(\dots, h_{-1}, h_0, h_1, \dots).$$

If the filter is algebraic and has finite impulse response, then the minimal necessary field is a number field, i. e., has finite dimension when interpreted as a vector space over the rationals.

Denote by $\overline{K}$ the normal closure[1] of the minimal necessary field. A scaling filter $(h_0, \dots, h_{N-1})$ is said to have the *conjugacy property* iff there exists an automorphism σ of the Galois group $\mathrm{Gal}(\overline{K}/\mathbb{Q})$ such that

$$(\sigma h_0, \dots, \sigma h_{N-1}) = (h_{N-1}, \dots, h_0). \tag{3}$$

Writing the scaling coefficients with respect to a base B of the field extension $K/\mathbb{Q}$ yields the scaling coefficients in vector form $(m_0, \dots, m_{N-1})$. Interpreting this vector with respect to the conjugate base σB leads to a representation of the "mirrored" scaling filter (3). Thus, we can implement every algebraic wavelet QMF pair that satisfies the conjugacy property as in Fig. 1.

[1] The normal closure of K is the field generated by all conjugate fields of K.

D. Back Conversion

In the previous example the results a_k and d_k are represented with respect to the bases B and σB, that is, the vector $a_k = (a_{k,1}, a_{k,2}) \in \mathbb{Q}^2$ represents the number

$$(a_{k,1}, a_{k,2})\ \left(1, \sqrt{3}\right)^T = a_{k,1} + \sqrt{3}\, a_{k,2}$$

and the vector $d_k = (d_{k,1}, d_{k,2}) \in \mathbb{Q}^2$ represents the number

$$(d_{k,1}, d_{k,2})\ \left(1, -\sqrt{3}\right)^T = d_{k,1} - \sqrt{3}\, d_{k,2}.$$

A rational approximation to these numbers is used in finite precision implementations. This can be achieved by rounding the elements of the bases B and σB to the desired precision, for instance, rounding the elements of B to three decimal digits yields $\{1, 1.732\}$. Thus, an implementation of the back conversion unit CVT in Fig. 1 requires at most four multiplications with rational constants and two additions, provided K is a quadratic field extension of the rationals.

E. Hardware issues

The growing significance of wavelet applications and their VLSI implementations were the main motivation for this research. We briefly explain the benefit of the proposed method in the case of the Daubechies 4-tab filters. Most applications require only a fixed filter pair. Therefore, fully-fledged multipliers can be avoided using hardwired adders, subtractors, and shifts, saving costly silicon area. We use the number of adders/subtractors (denoted by AS) as a complexity measure.

A naive implementation obtained by rounding each coefficient to eight binary digits after the fixed point can be realized with 26 AS. The method described in section III-B yields 18 AS for the same accuracy, assuming that the coefficients are expressed with respect to base B. Choosing another base of the field extension may lead to lower complexity implementations. For example, expressing the coefficients with respect to the base $\{1/2, (1+\sqrt{3})/8\}$ yields 14 AS. Note that the output of the dashed box in Fig. 1 changes only every second clock cycle. Therefore, it is possible to use the back conversion hardware jointly for the bases B and σB, leading to quite pronounced savings. In fact, it is then possible to realize the Daubechies filter with 9 AS in total. Implementation details and a comparison of layouts will be treated in a forthcoming paper [5].

IV. Density

The Daubechies wavelet of order two served as an example of an algebraic wavelet that satisfies the conjugacy property. One might wonder if there are more wavelets that satisfy these contrived conditions. The affirmative answer is given by the following theorem:

Theorem 1: All wavelet QMF filters derived from compactly supported orthonormal wavelet bases can be approximated with arbitrary precision by algebraic wavelet QMF filters that have the conjugacy property. Moreover, the minimal necessary field for these filters can be chosen of degree two.

We need a handsome description of all wavelet QMFs to prove this theorem. There exist several parametrizations of QMFs [6], [7], [8], [9]. It turns out that the parametrization due to D. Pollen [7] is well-suited for our purposes. We review the necessary facts of this parametrization in the next subsection.

A. Pollen's Parametrization

Pollen associates to every scaling filter a 2×2-matrix with Laurent polynomials in $\mathbb{R}[z, 1/z]$ as entries. The scaling coefficient constraints imply that the associated matrices are unitary, have determinant one, and yield the identity matrix when

1 is substituted for z. These matrices constitute a group called $SU_I(2,\mathbb{R}[z,1/z])$. This group is generated by a set of simply structured matrices (described below), leading to the desired parametrization of scaling filters.

Let us equip the ring of Laurent polynomials $\mathbb{R}[z,1/z]$ with the involutory operation $\widetilde{\ }$ given by $\widetilde{p}(z) := p(1/z)$. A scaling filter $H(z) = H_{Even}(z^2) + z^{-1}H_{Odd}(z^2)$ can be represented by the matrix

$$E(z) := \begin{pmatrix} u(z) & v(z) \\ -\widetilde{v(z)} & \widetilde{u(z)} \end{pmatrix}, \tag{4}$$

regarding the abbreviations $u(z) = \widetilde{H}_{Even}(z) + H_{Odd}(z)$ and $v(z) = -H_{Even}(z) + \widetilde{H}_{Odd}(z)$. Elementary calculations show that this representation is faithful, namely

$$H(z) = (-1/2, 1/2)\, E(z^2)\, (-z^{-1}, 1)^T. \tag{5}$$

Moreover, the mapping $H(z) \mapsto E(z)$ is a bijection between the set of scaling coefficients and the infinite dimensional Lie group $SU_I(2,\mathbb{R}[z,1/z])$. The inverse of an group element $E(z)$ is given by $E^{-1}(z) = \widetilde{E}(z)^T$. The group $SU_I(2,\mathbb{R}[z,1/z])$ can be generated by matrices of the form

$$U_\theta(z) := \begin{pmatrix} u_\theta(z) & v_\theta(z) \\ -\widetilde{v_\theta(z)} & \widetilde{u_\theta(z)} \end{pmatrix}, \tag{6}$$

where $u_\theta(z)$ and $v_\theta(z)$ are defined as follows:

$$\begin{aligned} u_\theta(z) &:= \frac{1}{2}\left[(1-\cos\theta)\, z + (1+\cos\theta)\right], \\ v_\theta(z) &:= \frac{1}{2}\left[(-\sin\theta) + (\sin\theta)\, z^{-1}\right]. \end{aligned}$$

Recall that every scaling coefficient sequence is of even length, because of the scaling coefficient constraint (1). Pollen's parametrization theorem essentially states that every scaling coefficient sequence of length N or less can be represented by a product of $N/2-1$ matrices of the form $U_\theta(z)$ or $U_\theta^{-1}(z)$.

More precisely, a scaling coefficient sequence of length $N = 4k+2$, $k \in \mathbb{N}$, can be represented by a product of the form

$$E(z) = U_{\theta_1} U_{\theta_2}^{-1} \cdots U_{\theta_{2k-1}} U_{\theta_{2k}}^{-1}, \tag{7}$$

and a scaling coefficient sequence of length $N = 4k+4$, $k \in \mathbb{N}$, can be represented by a product of the form

$$E(z) = U_{\theta_1} U_{\theta_2}^{-1} \cdots U_{\theta_{2k-1}} U_{\theta_{2k}}^{-1} U_{\theta_{2k+1}}. \tag{8}$$

For example, all scaling coefficient sequences of length four or less are given by

$$\begin{aligned} h_0(\theta) &= -1/4\,\cos\theta + 1/4\,\sin\theta + 1/4, \\ h_1(\theta) &= 1/4\,\cos\theta + 1/4\,\sin\theta + 1/4, \\ h_2(\theta) &= 1/4\,\cos\theta - 1/4\,\sin\theta + 1/4, \\ h_3(\theta) &= -1/4\,\cos\theta - 1/4\,\sin\theta + 1/4. \end{aligned} \tag{9}$$

Translating a scaling filter $H(z)$ by z^j leads again to a scaling filter $z^j\, H(z)$. These shifts are ignored in the parametrization of scaling coefficients.

B. A Symmetry of Pollen's Parametrization

Our goal is to construct algebraic wavelets satisfying the conjugacy property with the help of Pollen's parametrization. In this subsection we show how the parameters of the mirrored scaling sequence can be obtained from the parameters of the scaling sequence. We need the following technical result to simplify our notation.

Lemma 2: Denote by $\mathrm{Mat}_2(R)$ the ring of 2×2-matrices over a ring R. The operation M defined by

$$M\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

is an involutory automorphism of $\mathrm{Mat}_2(R)$. In particular, the relation $M(A)M(B) = M(AB)$ holds for all $A, B \in \mathrm{Mat}_2(R)$.

Proof: By inspection. □

In the previous example (9) it is easily seen that a sign change of the parameter θ leads to the mirrored scaling sequence. More generally, the following lemma describes a representation of the mirrored scaling filter.

Lemma 3: If $E(z)$ is the representation matrix (4) of a scaling filter $H(z)$, then $M(E(z))$ is a representation of the filter $z^{-1}\widetilde{H}(z)$.

Proof: Straightforward calculation, substitute the matrix $M(E(z^2))$ for $E(z^2)$ in (5). □

The operation of M on a matrix U_θ of the form (6) can be described in terms of the parameter θ by

$$M(U_\theta(z)) = U_{-\theta}(z). \tag{10}$$

This follows directly from the fact that $\sin\theta$ is an odd function and $\cos\theta$ is an even function. Similarly, the action on the inverse U_θ^{-1} is given by

$$M(U_\theta^{-1}(z)) = M(\widetilde{U}_\theta^T(z)) = \widetilde{U}_{-\theta}^T(z) = U_{-\theta}^{-1}(z). \tag{11}$$

Equations (10) and (11) together with Lemma 2 allow us to prove by induction that the mirrored scaling filter is obtained by a sign change of the parameters θ_i. Summarizing, we obtained the following result.

Proposition 4: Given a scaling filter $H(z)$ with representation matrix $E(z)$ of the form (7) or (8), then the representation matrix $M(E(z))$ of the mirrored scaling filter $z^{-1}\widetilde{H}(z)$ is obtained by sign change of all parameters θ_i.

C. Proof of Theorem 1

Before concluding the proof of Theorem 1, we illustrate the style of argument for scaling coefficient sequences of length four or less.

C.1. Illustration. The unit circle can be viewed as an affine variety defined by the equation $x^2 + y^2 = 1$. The circle can be parametrized by the trigonometric functions $x = \cos\theta$ and $y = \sin\theta$, or alternatively by the rational parametrization

$$x = \frac{1-\nu^2}{1+\nu^2}, \quad y = \frac{2\nu}{1+\nu^2}.$$

The last parametrization covers all points of the circle, except a single point $(-1,0)$. In view of this parametrization, example (9) can be re-expressed as follows:

$$\begin{aligned} h_0(\nu) &= \frac{1}{2}\frac{\nu(1+\nu)}{(1+\nu^2)}, & h_1(\nu) &= \frac{1}{2}\frac{(1+\nu)}{(1+\nu^2)}, \\ h_2(\nu) &= \frac{1}{2}\frac{(1-\nu)}{(1+\nu^2)}, & h_3(\nu) &= \frac{1}{2}\frac{\nu(\nu-1)}{(1+\nu^2)}. \end{aligned} \tag{12}$$

Each coefficient $h_i(\nu)$ depends continuously on the parameter ν. As a consequence the scaling filters of length four or less depend continuously on this parameter.

Let d be an arbitrary non-square rational. The rational multiples of $\sqrt{d}$ are dense in $\mathbb{R}$ and each parameter $\nu \in \sqrt{d}\,\mathbb{Q}$ yields an algebraic scaling filter that satisfies the conjugacy property. This shows that Theorem 1 holds for scaling filters of length four or less.

C.2. General Case. Replace the trigonometric functions in (6) by their rational parametrizations. This leads to a matrix $U_\nu(z)$ of the form (6), where u_ν and v_ν are defined by

$$\begin{aligned} u_\nu(z) &:= \left(\frac{\nu^2}{1+\nu^2}\right) z + \left(\frac{2}{1+\nu^2}\right), \\ v_\nu(z) &:= -\left(\frac{\nu}{1+\nu^2}\right) + \left(\frac{\nu}{1+\nu^2}\right) z^{-1}. \end{aligned}$$

Thus, the matrices $U_\nu(z)$ and $U_\nu^{-1}(z) = \widetilde{U}_\nu^T(z)$ have Laurent polynomials with coefficients in $\mathbb{Q}(\nu)$ as entries. A product of such matrices

$$E(z) = V_{\nu_1} V_{\nu_2} \cdots V_{\nu_K},$$

where $V_{\nu_i} \in \left\{U_{\nu_i}(z), U_{\nu_i}^{-1}(z)\right\}$, has Laurent polynomials with coefficients in $\mathbb{Q}(\nu_1, \ldots, \nu_K)$ as entries. Moreover, the denominator of each such coefficient is non-zero (by construction) for all choices of parameters. This shows that the scaling coefficient sequences of a given length, say length $2K+2$, depend continuously on the parameters $\nu_1, \ldots, \nu_K$.

Therefore, it is enough to show that there exists a dense set D in the parameter space, such that all parameters in D lead to algebraic scaling filters satisfying the conjugacy property.

Let d be a non-square rational. The set $D = (\sqrt{d}\,\mathbb{Q})^K$ is dense in $\mathbb{R}^K$. Apparently, all parameters $(\nu_1, \ldots, \nu_K) \in D$ lead to algebraic scaling filters. In fact, all scaling coefficient are elements of the quadratic number field $\mathbb{Q}(\sqrt{d})$. The action of the automorphism $\sigma : \sqrt{d} \mapsto -\sqrt{d}$ on a factor V_{ν_i} is equivalent to the action of the operator M:

$$\sigma V_{\nu_i} = M(V_{\nu_i}) = V_{-\nu_i}.$$

Using the homomorphy of σ and Proposition 4, it follows that

$$\begin{aligned} \sigma E(z) &= \sigma V_{\nu_1} \sigma V_{\nu_2} \cdots \sigma V_{\nu_K} \\ &= V_{-\nu_1} V_{-\nu_2} \cdots V_{-\nu_K} \quad = \quad M(E(z)). \end{aligned}$$

Thus, all scaling filters corresponding to the dense set of parameters D lead to algebraic scaling filters satisfying the conjugacy property. All these filters afford a minimal necessary field extension of degree two. □

V. Concluding Remarks

Orthonormal wavelet coefficients can have a quite complicated arithmetic structure. Indeed, it follows from example (12) that all real algebraic number fields can occur as minimal necessary fields. An algebraic wavelet filter affording a number field of high degree would jeopardize any effort to save silicon area using the methods described in section III. Theorem 1 shows that it is possible to circumvent this problem by approximating the desired wavelet QMF with simply structured filters.

The filters derived from compactly supported orthonormal wavelets can not be symmetric (with exception of the Haar filter). However, it was shown in this note that deeper symmetries can be exploited to reduce the complexity of an implementation.

Acknowledgements

I thank Professor Thomas Beth, Volker Baumgarte, Armin Nückel, and Frank May for many pleasant discussions.

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