On Some Limits of Lattice and Lifting Structures

Andreas Klappenecker

Department of Mathematics, Texas A&M University
College Station, TX, 77843-3368, USA
andreask@math.tamu.edu

ABSTRACT

We discuss the relation between lattice and ladder structures for two-channel filter banks. It is well-known that both lattice and ladder steps are powerful enough to generate all perfect reconstructing filter banks provided that the filter coefficients may take arbitrary values in a field. However, we will show that the two concept differ in general. We relate the two concepts by looking at three properties of the coefficient ring. We discuss a number of incompleteness results of these parametrizations and point out some connections to open problems in group theory.

1. INTRODUCTION

Assume that a signal is given by an element of the Laurent polynomial ring $A[z, z^{-1}]$, where A is either the field of real numbers $A = \mathbf{R}$ or the field of complex numbers $A = \mathbf{C}$. Recall that a multirate filter bank basically computes the convolution of this signal with several analysis filters, and reduces the sampling rate, say, by dropping every other output coefficient. Figure 1 shows an example of such a filter bank.

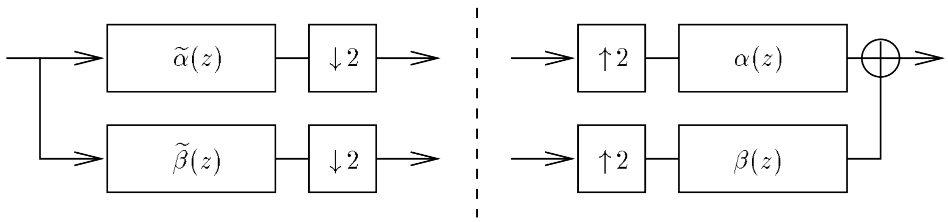


Figure 1. Two-channel filter bank.

Define the downsampling operation by $[\downarrow 2]a(z) = a_e(z)$, where $a(z) = a_e(z^2) + z a_o(z^2)$. Assuming that the analysis filters $\widetilde{\alpha}(z)$ and $\widetilde{\beta}(z)$ are element of $A[z, z^{-1}]$ as well, we can describe the output of the analysis filter bank by $d_\alpha(z) := [\downarrow 2]\widetilde{\alpha}(z)s(z)$ and $d_\beta(z) := [\downarrow 2]\widetilde{\beta}(z)s(z)$.

The synthesis filter bank takes two input signals $d_\alpha(z)$ and $d_\beta(z)$, applies upsampling and convolution operations, and adds the resulting sequences. We obtain

$$\hat{s}(z) = \alpha(z)([\uparrow 2]d_\alpha(z)) + \beta(z)([\uparrow 2]d_\beta(z)) = \alpha(z)d_\alpha(z^2) + \beta(z)d_\beta(z^2),$$

where the upsampling operation $[\uparrow 2]$ is defined by $[\uparrow 2]a(z) = a(z^2)$.

Note that only addition and multiplication in A are needed to compute the convolution operation. Therefore, one might replace the real or complex number arithmetic for example by integer arithmetic $A = \mathbf{Z}$, or by finite field arithmetic $A = \mathbf{F}_q$. In fact, we may take for A any commutative ring. This broader viewpoint also gives a wider range of applications. For example, filter banks over finite rings have applications in error control coding, see [1] for this connection.

We will mainly study two-channel filter banks for one-dimensional signals in this paper. Since we allow rather general coefficient rings A, it should be pointed out that many results are also of relevance in the multidimensional setting. Some introductions to filter banks prefer to treat the two-channel case first, and then the "more difficult" case of filter banks with more channels. There is nothing wrong with this approach in the case of fields. However, I hope it will get clear from the following that the "many channel" case is in fact simpler than the two-channel case.

Notation. Henceforth we shall denote by z an indeterminate over the ring A. Let G be a group. We write $H \leq G$ in case H is a subgroup of G, and $H < G$ if H is a proper subgroup.

2. LADDER AND LATTICE STRUCTURES

Recall that each filter bank can be expressed in polyphase form.[2–4] Writing the input signal $s(z)$ in the form $s(z) = s_e(z^2) + z s_o(z^2)$, the output of the analysis filter bank can be expressed by

$$\begin{pmatrix} d_\alpha(z) \\ d_\beta(z) \end{pmatrix} = H_p \begin{pmatrix} s_e(z) \\ s_o(z) \end{pmatrix}, \quad \text{with} \quad H_p := \begin{pmatrix} \widetilde{\alpha}_e(z) & \widetilde{\alpha}_o(z) \\ \widetilde{\beta}_e(z) & \widetilde{\beta}_o(z) \end{pmatrix}, \tag{1}$$

where $\widetilde{\alpha}(z) = \widetilde{\alpha}_e(z^2) + z^{-1}\widetilde{\alpha}_o(z^2)$ and $\widetilde{\beta}(z) = \widetilde{\beta}_e(z^2) + z^{-1}\widetilde{\beta}_o(z^2)$. A similar reasoning shows that the polyphase components of $\hat{s}(z)$ can be expressed by a product $G_p^t(d_\alpha(z), d_\beta(z))^t$. The polyphase implementation shown in Figure 2 follows those computations.

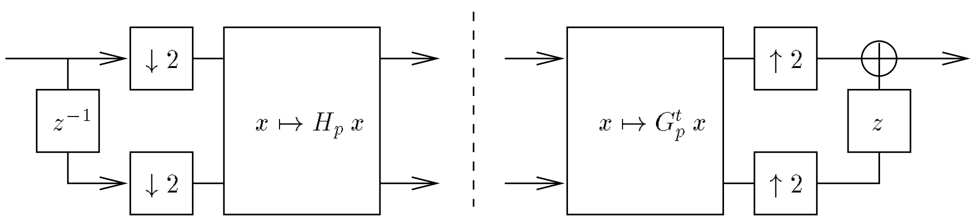


Figure 2. Polyphase implementation of a two-channel filter bank.

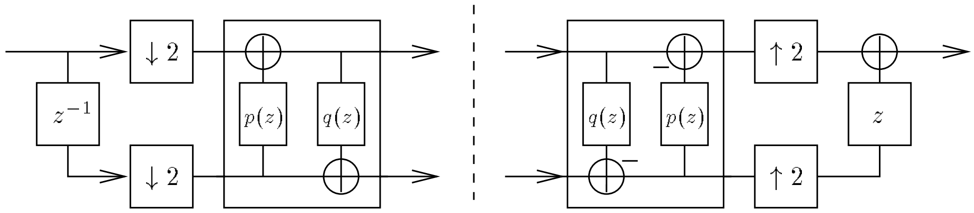
A filter bank is called perfect reconstructing if and only if the reconstructed signal $\hat{s}(z)$ is the same as the input signal $s(z)$ for all $s(z) \in A[z, z^{-1}]$. The filter bank shown in Figure 2 is perfect reconstructing if and only if the polyphase matrices satisfy the condition $H_p G_p^t = I$, where I is the 2×2 identity matrix.

The polyphase matrices are often factored into a product of certain simple matrices, which are easy to implement. The lattice and ladder structures are typical examples for this approach. These simple building blocks also serve a dual purpose, namely to give simple design criteria for filter banks.

Ladder structures. Ladder structures are composed of the following matrices in $\mathrm{GL}_2(A[z,z^{-1}])$: elementary transvections, that is, matrices of the type

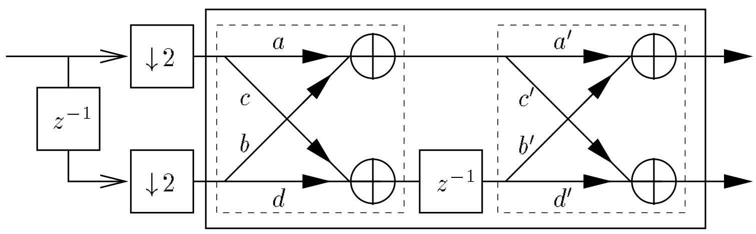
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \qquad a \in A[z,z^{-1}],$$

and invertible diagonal matrices. Why ladder structures bear their name can be seen from the following example:



We denote the subgroup of $\mathrm{GL}_2(A[z,z^{-1}])$ generated by the elementary transvections by $E_2(A[z,z^{-1}])$. The subgroup generated by $\mathrm{E}_2(A[z,z^{-1}])$ and the invertible diagonal matrices in $\mathrm{GL}_2(A[z,z^{-1}])$ is denoted by $\mathrm{GE}_2(A[z,z^{-1}])$.

Lattice structures. A matrix in $\mathrm{GL}_2(A[z,z^{-1}])$ is said to have a lattice structure if it can be written as a product matrices in $\mathrm{GL}_2(A)$ and powers of the diagonal delay matrix $\mathrm{diag}(1,z^{-1})$. The subgroup of $\mathrm{GL}_2(A[z,z^{-1}])$ generated by these matrices is called $\mathrm{GN}_2(A[z,z^{-1}])$. The next figure shows an example of lattice structures:



Of course, one would like to express all matrices in terms of these simple building blocks. Thus one would like to have the equalities:

$$\mathrm{GL}_2(A[z,z^{-1}]) = \mathrm{GE}_2(A[z,z^{-1}]) = \mathrm{GN}_2(A[z,z^{-1}]). \tag{2}$$

If the coefficient ring A is a field, then we indeed have the ideal situation described by the previous equations. Unfortunately, this is not true in general. In the next sections we discuss some relations between these three groups.

3. LIMITS OF LADDER AND LATTICE STRUCTURES

We need several terms from ring theory to describe properties of $A[z, z^{-1}]$. Recall that a ring is said to be reduced if and only if it does not contain non-zero nilpotent elements. A ring R is said to be indecomposable if and only if it does not contain central idempotent elements apart from 0 and 1. For example, the ring $\mathbf{Z}/4\mathbf{Z}$ is indecomposable but is not reduced; the nilpotent elements are $\{0, 2\}$. The ring $\mathbf{Z}/6\mathbf{Z}$ is reduced but not indecomposable; the idempotent elements are $\{0, 1, 3, 4\}$.

LEMMA 3.1. *Let A be a commutative ring, $B = A[z, z^{-1}]$.*

(i) $\mathrm{E}_2(B) < \mathrm{GN}_2(B)$.

(ii) If A is reduced and indecomposable, then $\mathrm{GE}_2(B) \leq \mathrm{GN}_2(B)$.

Proof. Let $a \in A$. The group $\mathrm{GN}_2(B)$ contains the matrices

$$\begin{pmatrix} 1 & az^k \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z^{-k} \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^k \end{pmatrix},$$

for all $k \in \mathbf{Z}$. Hence it contains all elementary transvections.

If A is reduced and incomposable, then $A[z, z^{-1}]$ contains only trivial units. Thus the invertible diagonal matrices are in $\mathrm{GN}_2(B)$. □

We need a further property of rings. A ring R is called a GE_2-ring if and only if $\mathrm{GE}_2(R) = \mathrm{GL}_2(R)$. If R is a field, a local ring, or a euclidean ring, then R is a GE_2-ring. Moreover, a finite product of GE_2-rings is again a GE_2-ring. Thus $\mathbf{Z}$, $\mathbf{Z}/N\mathbf{Z}$, $\mathbf{F}_p$, $\mathbf{R}$, and $\mathbf{C}$ are all examples of GE_2-rings.

PROPOSITION 3.2. *Let A be a GE_2-ring, $B = A[z, z^{-1}]$.*

Then $\mathrm{GN}_2(B) \leq \mathrm{GE}_2(B)$. *More precisely:*

(i) If A is decomposable or is not reduced, then $\mathrm{GN}_2(B) < \mathrm{GE}_2(B)$.

(ii) If A is indecomposable and reduced, then $\mathrm{GE}_2(B) = \mathrm{GN}_2(B)$.

Proof. By assumption, all matrices in $\mathrm{GL}_2(A)$ are contained in $\mathrm{GE}_2(A[z, z^{-1}])$. It follows that $\mathrm{GN}_2(B) \leq GE_2(B)$. Property (ii) follows immediately by Lemma 3.1.

If A is decomposable or is not reduced, then B contains non-trivial units. Since the determinant of a matrix in $\mathrm{GN}_2(B)$ is a trivial unit, the group $\mathrm{GN}_2(B)$ is a proper subgroup of $\mathrm{GE}_2(B)$. □

PROPOSITION 3.3. *Assume that A is not a GE_2-ring, $B = A[z, z^{-1}]$.*

(i) If A is reduced and indecomposable, then $\mathrm{GE}_2(B) < \mathrm{GN}_2(B)$.

(ii) If A is decomposable or is not reduced, then $\mathrm{GE}_2(B) \neq \mathrm{GN}_2(B)$.

Moreover, $\mathrm{GE}_2(B) < \mathrm{GL}_2(B)$ *and* $\mathrm{GN}_2(B) < \mathrm{GL}_2(B)$.

Proof. Part (i) is an immediate consequence of the assumption and Lemma 3.1.

If A is decomposable or is not reduced, then B contains non-trivial units. Thus, $\mathrm{GN}_2(B) < \mathrm{GL}_2(B)$. From the assumption it follows that $\mathrm{GE}_2(B) < \mathrm{GL}_2(B)$. Another consequence is that there is a matrix in $\mathrm{GN}_2(B)$ that is not in $\mathrm{GE}_2(B)$. This shows part (ii). □

We summarize the content of Lemma 3.1, Proposition 3.2, and Proposition 3.3 in Table 1.The case where the coefficient ring A is an indecomposable, reduced GE_2-ring seems to be particularly interesting. Here one has the flexibility to choose between ladder or lattice structures. Moreover, one can hope that equation (2) is satisfied – at least for nice coefficient rings A. We will have a closer look at such rings in the following sections.

A is:	GE_2	reduced	indecomp.	Lattice vs. Ladder	Comment
	no	no	no	$\mathrm{GN}_2(B) \neq \mathrm{GE}_2(B)$	both incomplete
	no	yes	yes	$\mathrm{GN}_2(B) \neq \mathrm{GE}_2(B)$	both incomplete
	no	no	no	$\mathrm{GN}_2(B) \neq \mathrm{GE}_2(B)$	both incomplete
	no	yes	yes	$\mathrm{GN}_2(B) > \mathrm{GE}_2(B)$	ladder incomplete
	yes	no	no	$\mathrm{GN}_2(B) < \mathrm{GE}_2(B)$	lattice incomplete
	yes	yes	yes	$\mathrm{GN}_2(B) < \mathrm{GE}_2(B)$	lattice incomplete
	yes	no	no	$\mathrm{GN}_2(B) < \mathrm{GE}_2(B)$	lattice incomplete
	yes	yes	yes	$\mathrm{GN}_2(B) = \mathrm{GE}_2(B)$	

Table 1. The relation between lattice and ladder structures.

4. INCOMPLETENESS RESULTS

Unfortunately, it is not clear what kind of restrictions should be imposed on the coefficient ring to guarantee that the desired equalities (2) hold. Since even principal ideal domains may fail to be GE_2-rings, one can not expect an easy answer. In this section we discuss a number of incompleteness results, that is, examples of perfect reconstructing filter banks that can be neither expressed by lattice nor ladder structures. As suggested in the previous section, we will focus on reduced and indecomposable coefficient rings.

An integral domain is in particular an indecomposable and reduced ring. There are a number of integral domains that are GE_2-rings: any field, or any euclidean domain.

Theorem 4.1. *Let A be an integral domain which is not a field, and x an indeterminate over A. Then the polynomial ring $A[x]$ is not a* GE_2*-ring.*

An elementary proof of this fact can be found in [5]. It is clear from this theorem that we can not hope to find many examples of integral domains A that yield the desired completeness result (2). In particular, if k is a field, then $A = k[x]$ is an euclidean domain. But even for such a nice coefficient ring, we see that $A[z, z^{-1}] \cong k[x][z, z^{-1}] \cong k[z, z^{-1}][x]$ is not a GE_2-ring by the previous theorem. In a similar vein, the following theorem also excludes numerous Laurent polynomial rings:

Theorem 4.2 (Bachmuth, Mochizuki[6]). *Let $A = P[t, t^{-1}]$, where P is an integral domain which is not a field, and t is an indeterminate over P. Then $A[z, z^{-1}]$ is not a* GE_2*-ring.*

Note that there is a remarkable gap between the statements of Theorem 4.1 and Theorem 4.2. Let A be an integral domain which is not a field. Theorem 4.1 states that all polynomial rings over A are not GE_2-rings. From Theorem 4.2 one can only deduce that Laurent polynomial rings in at least two variables over A are not GE_2-rings. Thus, there is some hope that a Laurent polynomial ring over, say, some euclidean or more generally some Dedekind ring is a GE_2-ring. We have already seen that this is not always the case: the euclidean ring $A = k[x]$, k a field, provides a counter example.

A discrete valuation ring is a principal ideal domain that has a unique non-zero prime ideal. Let p be a prime in $\mathbf{Z}$. The localization $\mathbf{Z}_{(p)}$ of the integers $\mathbf{Z}$ at the prime ideal (p) is an example of a discrete valuation ring. In other words, $\mathbf{Z}_{(p)}$ is given by the subset of the rational numbers consisting of fractions a/b, where b is not divisible by p. We have the following positive answer for this special class of Dedekind rings:

THEOREM 4.3 (BACHMUTH, MOCHIZUKI[6]). *Let A be a discrete valuation ring. Then $A[z, z^{-1}]$ is a* GE_2*-ring.*

In any noetherian ring one tends to globalize the results. However, the GE_2-property is resistant against such local-global principles. In fact, the localization of a Dedekind ring at a prime ideal gives a discrete valuation ring. However, we have already seen that there exist Laurent polynomial rings over Dedekind domains that are not GE_2-rings. That those negative examples abound is shown in the next section.

5. MANY CHANNELS

The two-channel case is rather pathological in the sense that there are only two different types of ladder steps available. In this section we allow an arbitrary number of channels. This simplifies the discussion, since there are a number of powerful tools available from Algebraic K-Theory, a branch of Linear Algebra.

Consider a perfect reconstructing filter bank with n channels. Assume that the downsampling operator keeps 1 out of n coefficients. Assume further that the signals and filters are elements of $A[z, z^{-1}]$. If the filter bank is perfect reconstructing, then the polyphase matrix H_p of the analysis filter bank is an element of $\mathrm{GL}_n(A[z, z^{-1}])$.[7] We would like to know if it is possible to implement all such perfect reconstructing filter banks with ladder steps.

Let us fix some terminology. An elementary transvection in $\mathrm{GL}_n(A[z, z^{-1}])$ is a matrix that differs from the identity matrix in at most one off-diagonal entry. The group generated by elementary matrices is called $\mathrm{E}_n(A[z, z^{-1}])$. $\mathrm{GE}_n(A[z, z^{-1}])$ denotes the group generated by $\mathrm{E}_n(A[z, z^{-1}])$ and the invertible diagonal matrices. We can reformulate our question as follows: does the equality $\mathrm{GL}_n(A[z, z^{-1}]) = \mathrm{GE}_n(A[z, z^{-1}])$ hold? Note that this question is equivalent to the following: does the equality $\mathrm{SL}_n(A[z, z^{-1}]) = \mathrm{E}_n(A[z, z^{-1}])$ hold?

We can obtain an answer to this question for certain coefficient rings, and large n, with methods from Algebraic K-Theory. Although the methods are somewhat technical, one is rewarded with surprisingly strong results. In the first step we recall the definition of the Whitehead group $K_1(R)$, which measures in some sense the obstruction to our question for large n.

Let R be a commutative ring. Identify a matrix $M \in \mathrm{GL}_n(R)$ with the block diagonal matrix $\mathrm{diag}(M, 1) \in \mathrm{GL}_{n+1}(R)$. Define the group $\mathrm{GL}(R)$ as the direct limit

$$\mathrm{GL}(R) = \varinjlim \mathrm{GL}_n(R) = \bigcup_{i=1}^{\infty} \mathrm{GL}_n(R).$$

Similarly, we put $\mathrm{E}(R) = \bigcup_n E_n(R)$ and $\mathrm{SL}(R) = \bigcup_n SL_n(R)$.

The Whitehead group $K_1(R)$ is defined by $\mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$, the quotient of $\mathrm{GL}(R)$ by its commutator subgroup. It turns out that $[\mathrm{GL}(R), \mathrm{GL}(R)]$ coincides with $\mathrm{E}(R)$. The determinant of each matrix provides us with an epimorphism from $K_1(R)$ to the unit group $R^\times$ of the ring R. We obtain the following short exact sequence:

$$1 \longrightarrow \underbrace{\mathrm{SL}(R)/\mathrm{E}(R)}_{=SK_1(R)} \longrightarrow K_1(R) \longrightarrow R^\times \longrightarrow 1.$$

The determinant homomorphism is split by $R^\times \longrightarrow \mathrm{GL}_1(R)$. Therefore, the Whitehead group decomposes

$$K_1(R) = SK_1(R) \oplus R^\times.$$

The calculation of the Whitehead group provides us with valuable information for our problem, since there exists the following stability result. Let R be a noetherian ring with finite Krull dimension $\dim(R) = d$. For all $m \geq d+2$, $\mathrm{E}_m(R)$ is a normal subgroup of $\mathrm{GL}_m(R)$, and $\mathrm{GL}_m(R)/\mathrm{E}_m(R) \to K_1(R)$ is an isomorphism.

A ring R is said to be regular if R is noetherian and all finitely generated R-modules have a projective resolution of finite type.[8] For example, any field or any Dedekind ring is regular. If A is regular, then so is the polynomial ring $A[z]$ and the Laurent polynomial ring $A[z,z^{-1}]$.[9,10] Moreover, for regular rings $K_1(A[z]) \cong K_1(A)$ and $K_1(A[z,z^{-1}]) \cong K_1(A) \oplus K_0(A)$, where $K_0(A)$ denotes the Grothendieck group.

Recall that our original question was whether or not we have $\mathrm{SL}_n(A[z,z^{-1}]) = E_n(A[z,z^{-1}])$. We show how an answer can be obtained for Dedekind rings. Why I focus on this particular case will be justified later.

Let D be a Dedekind ring. The Krull dimension is $\dim(D) \leq 1$, hence $\dim(D[z,z^{-1}]) \leq 2$. We obtain

$$SK_1(D[z,z^{-1}]) \cong SK_1(D) \oplus K_0(D)/\mathbf{Z}[D],$$

where $[D]$ denotes the image of the isomorphism class of D in the Grothendieck group. For example, if D is the ring of integers in an algebraic number field or if D is an euclidean ring, then $SK_1(D) = 1$. The reduced Grothendieck group $K_0(D)/\mathbf{Z}[D]$ is isomorphic to the class group of the Dedekind ring D.[11]

We have $\mathrm{SL}_n(D[z,z^{-1}]) = E_n(D[z,z^{-1}])$ for $n \geq 4$ if and only if $SK_1(D) = 1$ and the class group of D is trivial. Those conditions are satisfied, for example, if D is a euclidean ring. In fact, the stability bound $n \geq 4$ can be lowered to $n \geq 3$ by the following result of Suslin[12]:

Theorem 5.1 (Suslin). *Let A be a noetherian ring, and*

$$B = A[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}, x_{k+1}, \ldots, x_n].$$

Then, for all $r \geq \max(3, \dim A + 2)$, the canonical mapping $\mathrm{GL}_r(B)/E_r(B) \to K_1(B)$ is an isomorphism.

Remark. In view of Suslin's theorem, it is clear why I have chosen coefficient rings of Krull dimension ≤ 1. Here we can derive results for filter banks with as few as three channels. An indecomposable, regular ring is a noetherian, integrally closed domain.[13] A noetherian, integrally closed domain of Krull dimension ≤ 1 is a Dedekind ring. This explains why I focused on this particular case. Dropping the regularity condition typically leads to more involved calculations.[9]

6. CONCLUSION

We have seen that in the two-channel it is often impossible to realize certain filter banks with ladder of lattice structures. Negative examples are even given by nice coefficient rings, such as euclidean domains. A remarkable open problem seems to be the following question, which was raised by Bachmuth and Mochizuki 18 years ago:

$$\text{Is } \mathbf{Z}[z, z^{-1}] \text{ a } \mathrm{GE}_2\text{-ring?}$$

Apparently, there has been no progress on this question since then. The related question for causal filter banks with minimum delay has a negative answer.[14,15,5]

REFERENCES

1. A. Klappenecker, "On multirate filter bank structures." To appear in Proc. of 2nd Workshop on Transforms and Filter Banks, Brandenburg, 1999.
2. G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, Wellesley, 1996.
3. P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall, 1993.
4. M. Vetterli and J. Kovačević, *Wavelets and Subband Coding*, Prentice Hall, 1995.
5. A. Klappenecker, M. Holschneider, and K. Flornes, "Two-channel perfect reconstruction filter banks over commutative rings." To appear in *Appl. Comp. Harm. Anal.*, 1999.
6. S. Bachmuth and H. Y. Mochizuki, "$E_2 \neq SL_2$ for most Laurent polynomial rings," *Amer. J. Math.* **104**(6), pp. 1181–1189, 1982.
7. A. Klappenecker and M. Holschneider, "A unified view on filter banks," in *Wavelet Applications in Signal and Image Processing VI*, A. F. Laine, M. A. Unser, and A. Aldroubi, eds., vol. 3458, pp. 2–13, SPIE, 1998.
8. J. Rosenberg, *Algebraic K-Theory and Its Applications*, Springer Verlag, New York, 1994.
9. H. Bass, *Algebraic K-Theory*, Benjamin, 1968.
10. H. Bass, *Introduction to some methods of algebraic K-theory*, vol. 20 of *CBMS Regional conference series in mathematics*, American Mathematical Society, 1974.
11. J. Milnor, *Introduction to Algebraic K-Theory*, vol. 72 of *Ann. Math. Studies*, Princeton Univ. Press, 1971.
12. A. A. Suslin, "On the structure of the special linear group over polynomial rings," *Izv. Akad. Nauk SSSR* **41**(2), pp. 221–238, 1977.
13. R. Brüske, F. Ischebeck, and F. Vogel, *Kommutative Algebra*, BI Wissenschaftsverlag, Mannheim, 1989.
14. P. M. Cohn, "On the structure of the GL_2 of a ring," *Publ. math. I.H.E.S.* **30**, pp. 365–413, 1966.
15. L. Tolhuizen, H. Hollmann, and A. Kalker, "On the realizability of bi-orthogonal m-dimensional 2-band filter banks," *IEEE Trans. on Signal Proc.* **43**, pp. 640–648, 1995.