Two-Channel Perfect Reconstruction FIR Filter Banks over Commutative Rings

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Abstract

The relation between ladder and lattice implementations of two-channel filter banks is discussed and it is shown that these two concepts differ in general. An elementary proof is given for the fact that over any integral domain which is not a field there exist causal realizable perfect reconstructing filter banks that can not be implemented with causal lifting filters. A complete parametrization of filter banks with coefficients in local rings and semiperfect rings is given.

List of Symbols

$\mathbf{Q}$	rational numbers
$\mathbf{Z}$	integers
$\mathbf{Z}/N\mathbf{Z}$	integers modulo N
A	a commutative ring with identity
$A^{\times}$	group of units of the ring A
$A[z^{-1}]$	polynomial ring in z^{-1} over the commutative ring A
$A[z, z^{-1}]$	Laurent polynomial ring over the commutative ring A
$\deg f$	degree of the polynomial f
$a \nmid b$	a does not divide b
$\mathrm{cnc}(p, q)$	p can not cancel q
$\mathrm{HM} f$	head monomial of the (Laurent) polynomial f
$\mathrm{GL}_n(R)$	general linear group over the ring R
$\mathrm{SL}_n(R)$	special linear group over the ring R
$\mathrm{E}_n(R)$	group generated by elementary transvections
$\mathrm{GE}_n(R)$	group generated by elementary matrices
$\mathrm{GN}_n(R)$	generalized Nagao group
M^t	transpose of the matrix M
$[\downarrow 2]$	downsampling by factor two
$[\uparrow 2]$	upsampling operation

1 Filter Banks

Let A be a commutative ring. We assume that all signals and filters are elements of the Laurent polynomial ring $B = A[z, z^{-1}]$. Note that B is isomorphic to the group algebra $A[\mathbf{Z}]$. Therefore, we refer to the multiplication in B as a *convolution* or *filter* operation. We define a downsampling operation $[\downarrow 2]$ on B by $[\downarrow 2]\, a(z) = a_e(z)$, where $a(z) = a_e(z^2) + z a_o(z^2)$. An upsampling operation $[\uparrow 2]$ on B is defined by $[\uparrow 2]\, a(z) = a(z^2)$.

A two-channel filter bank consists of two parts. In the *analysis part* an input signal $s(z) \in B$ is filtered with two analysis filters $\widetilde{\alpha}(z), \widetilde{\beta}(z) \in B$, and the sampling rate of the resulting two signals is reduced by applying $[\downarrow 2]$. Thus we obtain the signals $d_\alpha(z) := [\downarrow 2]\, \widetilde{\alpha}(z)s(z)$ and $d_\beta(z) := [\downarrow 2]\, \widetilde{\beta}(z)s(z)$.

The *synthesis filter bank* takes two signals $d_\alpha(z)$ and $d_\beta(z)$ as input, applies the upsampling operation, filters these signals with synthesis filters $\alpha(z)$ and $\beta(z)$ respectively, and adds the resulting two signals. This step yields the output signal

$$\hat{s}(z) := \alpha(z)d_\alpha(z^2) + \beta(z)d_\beta(z^2).$$

The analysis and synthesis parts of a two-channel filter bank are sketched in Figure 1.

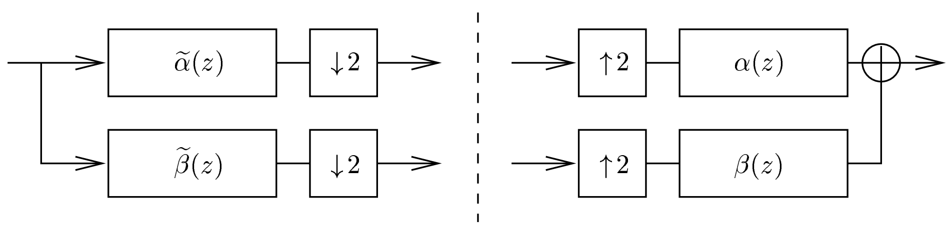

Figure 1: Two channel filter bank. The analysis (synthesis) filter bank is shown on the left (right) of the dashed line.

An equivalent but computationally more efficient implementation of such a filter bank can be derived as follows. Express the input signal $s(z)$ in the form $s(z) = s_e(z^2) + z s_o(z^2)$. The result of the *analysis filter bank* can be written in the form

$$\begin{pmatrix} d_\alpha(z) \\ d_\beta(z) \end{pmatrix} = H_p \begin{pmatrix} s_e(z) \\ s_o(z) \end{pmatrix}, \quad \text{with} \quad H_p := \begin{pmatrix} \widetilde{\alpha}_e(z) & \widetilde{\alpha}_o(z) \\ \widetilde{\beta}_e(z) & \widetilde{\beta}_o(z) \end{pmatrix}, \tag{1}$$

where $\widetilde{\alpha}(z) = \widetilde{\alpha}_e(z^2) + z^{-1}\widetilde{\alpha}_o(z^2)$ and $\widetilde{\beta}(z) = \widetilde{\beta}_e(z^2) + z^{-1}\widetilde{\beta}_o(z^2)$.

Similarly, if we express the output signal $\hat{s}(z)$ in the form $\hat{s}(z) = \hat{s}_e(z^2) + z\hat{s}_o(z^2)$, then the result of the *synthesis filter bank* can be written in the form

$$\begin{pmatrix} \hat{s}_e(z) \\ \hat{s}_o(z) \end{pmatrix} = G_p^t \begin{pmatrix} d_\alpha(z) \\ d_\beta(z) \end{pmatrix}, \quad \text{with} \quad G_p := \begin{pmatrix} \alpha_e(z) & \alpha_o(z) \\ \beta_e(z) & \beta_o(z) \end{pmatrix}, \qquad (2)$$

where $\alpha(z) = \alpha_e(z^2) + z\alpha_o(z^2)$ and $\beta(z) = \beta_e(z^2) + z\beta_o(z^2)$.

Figure 2 shows the implementation suggested by equations (1) and (2). In the signal processing literature this is known as the *polyphase form* of the filter bank [8, 11, 12].

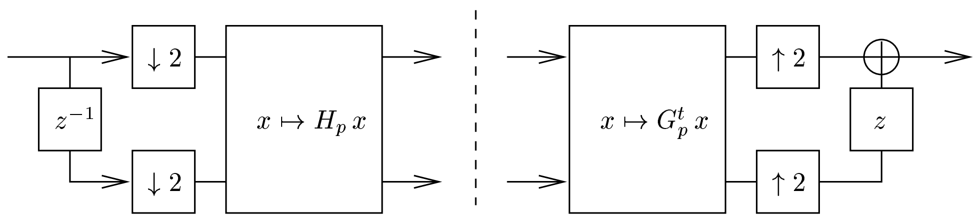

Figure 2: Two-channel filter bank in polyphase form.

A filter bank is said to be *perfect reconstructing*[1] if and only if the output signal $\hat{s}(z)$ coincides with the input signal $s(z)$ for all $s(z) \in B$. This condition can also be expressed as follows:

Theorem 1 *A two-channel filter bank for signals in $A[z, z^{-1}]$ is perfect reconstructing if and only if the polyphase matrices H_p and G_p satisfy the condition $G_p^t H_p = I$.*

Thus, the polyphase matrices of perfect reconstructing filter banks are elements of the general linear group $\mathrm{GL}_2(B)$. In some applications, e. g., lossless compression, the perfect reconstruction condition is a minimal requirement.

2 Ladder and Lattice Structures

In practice one wishes to design filter banks that are in some sense "easy" to implement. We review two methods that are widely used in the construction

[1] Sometimes this condition is relaxed and the output signal is only required to be a delayed version of the input signal.

of perfect reconstructing filter banks. Both methods use a product of simple matrices to construct the matrices H_p and G_p^t.

The basic building blocks of the first method are known in signal processing as *ladder structures* or *lifting steps* [10, 4, 2]. They are given by the elementary transvections

$$T_{12}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad T_{21}(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \tag{3}$$

with $a, b \in B$, and by diagonal matrices in $\mathrm{GL}_2(B)$. Figure 3 shows an example of such an implementation.

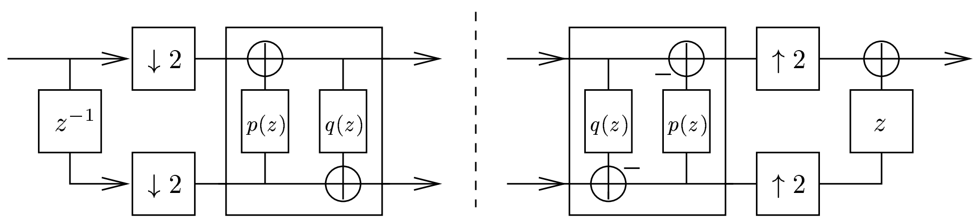

Figure 3: Ladder structures.

The group $\mathrm{GL}_2(B)$ of invertible 2×2-matrices has the three subgroups of elementary matrices

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \qquad \begin{pmatrix} B^\times & 0 \\ 0 & 1 \end{pmatrix},$$

where $B^\times$ denotes the set of units in B. The subgroup of $\mathrm{GL}_2(B)$ generated by these elementary matrices is called $\mathrm{GE}_2(B)$. Thus, the group $\mathrm{GE}_2(B)$ describes all polyphase matrices H_p and G_p^t that can be constructed with the help of some lifting filter network.

For later use we define the group $\mathrm{E}_2(B)$ to be the subgroup of $\mathrm{GE}_2(B)$ of matrices with determinant 1. Note that $\mathrm{E}_2(B)$ is generated by the elementary transvections.

The basic building blocks of the second method are known in signal processing as *lattice structures* [5, 11, 8]. In this case the matrices H_p and G_p^t are given as a product of constant matrices in $\mathrm{GL}_2(A)$ and powers of the delay matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

The subgroup in $\mathrm{GL}_2(B)$ generated by $\mathrm{GL}_2(A)$ and the delay matrix is called $\mathrm{GN}_2(B)$ here. The group $\mathrm{GN}_2(B)$ describes all polyphase matrices that can be generated by some lattice filter network.

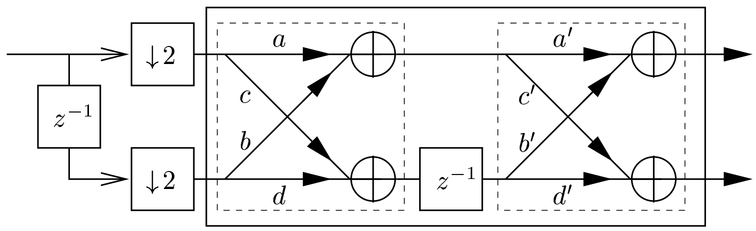

Figure 4: Analysis filter bank realized with lattice structures.

3 Relation between Ladder and Lattice Structures

If A is a field it is known that the ladder and the lattice approach are both powerful enough to generate all polyphase matrices in $\mathrm{GL}_2(B)$, cf. [7]. However, for more general coefficient rings the two concepts differ in general.

Recall that a ring is called reduced if and only if it does not contain non-zero nilpotent elements. A ring R is said to be indecomposable if and only if it does not contain central idempotent elements apart from 0 and 1. A ring R is called a GE_2-ring if and only if $\mathrm{GE}_2(R) = \mathrm{GL}_2(R)$.

Proposition 2 *Let A be a commutative ring, $B = A[z, z^{-1}]$.*

(i) $\mathrm{E}_2(B) \subset \mathrm{GN}_2(B)$.

(ii) If A is reduced and indecomposable, then $\mathrm{GE}_2(B) \subset \mathrm{GN}_2(B)$.

Proof. (i) The normalizer of

$$\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix}$$

in $\mathrm{GN}_2(B)$ contains

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

respectively, which shows that $\mathrm{E}_2(B) \subset \mathrm{GN}_2(B)$.

(ii) If A is reduced and indecomposable, then $A[z, z^{-1}]$ contains only trivial units, and therefore $\mathrm{GN}_2(B)$ contains the invertible diagonal matrices from $\mathrm{GL}_2(B)$. Using (i), this proves the claim. $\square$

Remarks.

1. If A is a subring of the field of complex numbers, then $\mathrm{GE}_2(B) \subset \mathrm{GN}_2(B)$. Thus, in this case all perfect reconstructing filter banks constructed with ladder steps can be constructed with the help of lattice steps as well. We will see below that the converse is not necessarily true.

2. If A is a field, then $A[z, z^{-1}]$ is a euclidean ring, and in particular a GE_2-ring. Therefore, $\mathrm{GE}_2(B) = \mathrm{GN}_2(B) = \mathrm{GL}_2(B)$, which shows the completeness of the lattice and ladder factorizations for fields.

Recall that a unit in B is called trivial if and only if it is of the form uz^k, for some $u \in A^\times$ and $k \in \mathbf{Z}$.

Proposition 3 *Let A be a GE_2-ring, $B = A[z, z^{-1}]$. Then $\mathrm{GN}_2(B) \subset \mathrm{GE}_2(B)$. More precisely:*

(i) If A is decomposable or is not reduced, then $\mathrm{GN}_2(B) \subsetneqq \mathrm{GE}_2(B)$.

(ii) If A is indecomposable and reduced, then $\mathrm{GE}_2(B) = \mathrm{GN}_2(B)$.

Proof. The matrices from $\mathrm{GL}_2(A)$ are contained in $\mathrm{GE}_2(B)$ by assumption. It is clear that the delay matrix is in $\mathrm{GE}_2(B)$. Therefore $\mathrm{GN}_2(B) \subset \mathrm{GE}_2(B)$.

We observe that the determinant of a matrix in $\mathrm{GN}_2(B)$ is a trivial unit.

(i) If A is decomposable, then A contains a non-trivial idempotent e. The element $u = e + (1-e)z^{-1}$ is a non-trivial unit in B. Thus, the elementary diagonal matrices $\mathrm{diag}(u, 1)$ is contained in $\mathrm{GE}_2(R)$ but not in $\mathrm{GN}_2(B)$.

If A is not reduced, then A contains a non-zero nilpotent element r. Thus, $v = 1 + rz^{-1}$ is a non-trivial unit in B. Again, the elementary diagonal matrix $\mathrm{diag}(v, 1)$ is contained in $\mathrm{GE}_2(R)$ but not in $\mathrm{GN}_2(B)$.

(ii) Clear from the above and Proposition 2. $\square$

Example 4 *If $A = \mathbf{Z}/N\mathbf{Z}$, where N is a positive integer that is not prime, then the previous proposition shows that $\mathrm{GN}_2(A[z, z^{-1}]) \subsetneqq \mathrm{GE}_2(A[z, z^{-1}])$. Thus we may construct filter banks with the help of ladder structures that can not be constructed with lattice structures.*

Proposition 5 *Assume that A is not a* GE_2*-ring,* $B = A[z, z^{-1}]$*.*

(i) If A is reduced and indecomposable, then $\mathrm{GE}_2(B) \subsetneqq \mathrm{GN}_2(B)$*.*

(ii) If A is decomposable or is not reduced, then $\mathrm{GE}_2(B) \neq \mathrm{GN}_2(B)$*. Moreover,* $\mathrm{GE}_2(B) \subsetneqq \mathrm{GL}_2(B)$ *and* $\mathrm{GN}_2(B) \subsetneqq \mathrm{GL}_2(B)$*.*

Proof. (i) By Proposition 2, $\mathrm{GE}_2(B) \subset \mathrm{GN}_2(B)$. Take a matrix $M \in \mathrm{GL}_2(A) \setminus \mathrm{GE}_2(A)$. Suppose that $M \in \mathrm{GE}_2(B)$. Then it is possible to factor M into a product of elementary matrices $M = \prod E_i(z)$. Specializing $z \mapsto 1$ would lead to a factorization $M = \prod E_i(1) \in \mathrm{GE}_2(A)$, contradicting our choice of M. This shows that $\mathrm{GN}_2(B) \setminus \mathrm{GE}_2(B)$ is not empty.

(ii) As in Proposition 3, we see that $B^\times$ contains non-trivial units. It follows that $\mathrm{GN}_2(B) \subsetneqq \mathrm{GL}_2(B)$ and $\mathrm{GE}_2(B) \neq \mathrm{GN}_2(B)$. Using the argument from (i), we can show that there exists a matrix $M \in \mathrm{GL}_2(A)$, which is not contained in $\mathrm{GE}_2(B)$. $\square$

Example 6 *Let A be the ring of integers in the imaginary quadratic number field* $\mathbf{Q}(\sqrt{-d})$*, where d is a positive, squarefree number,* $d \notin \{1, 2, 3, 7, 11\}$*. It was shown by Cohn [3] that A is not a* GE_2*-ring. It follows from the preceeding proposition that it is possible to construct filter banks with lattice factorization, say with filters in* $B = \mathbf{Z}[\sqrt{-5}][z, z^{-1}]$*, that can not be realized with ladder structures in* $\mathrm{GE}_2(B)$*.*

4 Incompleteness

An element in $B = A[z, z^{-1}]$ is called causal if it is already contained in $C = A[z^{-1}]$. A filter bank is said to be realizable if all its components are causal. We allowed a multiplication by z in the synthesis filter bank so far; this can be avoided using the delayed version of the filter bank as shown in Figure 5.

Such a filter bank is said to be perfect reconstructing if and only if the output signal $\hat{s}(z)$ coincides with the input signal $s(z)$ delayed by one, that is, $\hat{s}(z) = z^{-1}s(z)$ for all $s(z) \in B$. The filter bank in Figure 5 is realizable and perfect reconstructing if and only if $G_p^t H_p = I$ and $G_p^t, H_p \in \mathrm{GL}_2(C)$.

An application of the euclidean algorithm shows that $C = A[z^{-1}]$ is a GE_2-ring provided A is a field. In this case every $H_p \in \mathrm{GL}_2(C)$ can be factored into a product of elementary transvections $T_{12}(a), T_{21}(b)$, $a, b \in C$, and

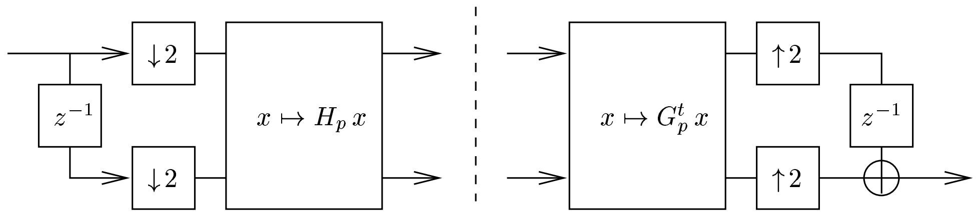

Figure 5: Delayed version of a two-channel filter bank.

diagonal matrices in $\mathrm{GL}_2(C)$. The purpose of this section is to give an elementary proof of the following incompleteness result:

Theorem 7 *Suppose that A is a non-trivial, commutative integral domain that is not a field, $C = A[z^{-1}]$. Then C is not a GE_2-ring.*

The head monomial $\mathrm{HM}(p)$ of a non-zero polynomial $p(z) = \sum_{i=0}^{n} p_i z^{-i}$, $p_n \neq 0$, is defined to be $\mathrm{HM}(p) = p_n z^{-n}$.

Let $p, q \in C$. We define the predicate $\mathrm{cnc}(p, q)$ to be true (and say that p can not cancel q) if and only if one of the following two conditions is true:

(a) $\deg(\mathrm{HM}(p)) = \deg(\mathrm{HM}(q))$ and there does not exist an element r of the quotient field $\mathrm{Quot}(A)$ such that $\mathrm{HM}(q) = r\mathrm{HM}(p)$.

(b) $\deg(\mathrm{HM}(p)) \neq \deg(\mathrm{HM}(q))$ and there does not exist an element $r \in C$ such that $\mathrm{HM}(q) = r\mathrm{HM}(p)$.

We say that a matrix $M \in \mathrm{SL}_2(C)$,

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

satisfies **condition S** if and only if the matrix entries p, q, r, and s are non-zero and $\mathrm{cnc}(p, q)$, $\mathrm{cnc}(q, p)$, $\mathrm{cnc}(r, s)$, and $\mathrm{cnc}(s, r)$ hold true.

Lemma 8 *Let C be as in Theorem 7. Let $y \in C$. If $M \in \mathrm{SL}_2(C)$ satisfies condition S, then $T_{21}(y)M$ satisfies condition S as well.*

Proof. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := T_{21}(y)M = \begin{pmatrix} p & q \\ py + r & qy + s \end{pmatrix}.$$

We know that a and b are non-zero and that

$$\mathrm{cnc}(a, b) \ \text{ and } \ \mathrm{cnc}(b, a) \quad \text{are true.} \tag{4}$$

1. Suppose that $c = 0$. Since $\det(T_{21}(y)M) = 1$, we have $1 = ad$, which implies $\mathrm{HM}(b) = \mathrm{HM}(abd) = \mathrm{HM}(a)\mathrm{HM}(bd)$, contradicting (4). Therefore, $c \neq 0$. A similar argument shows $d \neq 0$.

   We are left to show that $\mathrm{cnc}(c, d)$ and $\mathrm{cnc}(d, c)$ are true.

2. Suppose that $ad \in A$. Since $ad - bc = 1$, this implies that $bc \in A$. Since $cd \neq 0$, we derive that a and b are constants, contradicting (4). Therefore ad is not constant and hence bc is not constant.

3. Since $ad - bc = 1$ and ad and bc are not constant, the terms with largest total degree have to cancel. Therefore, we have

$$\mathrm{HM}(a)\mathrm{HM}(d) = \mathrm{HM}(ad) = \mathrm{HM}(bc) = \mathrm{HM}(b)\mathrm{HM}(c). \tag{5}$$

4. Seeking a contradiction, we suppose that $\mathrm{cnc}(c, d)$ is false.

   In the case $\deg(\mathrm{HM}(c)) \neq \deg(\mathrm{HM}(d))$ this means that there exists a polynomial $r \in C$ such that $\mathrm{HM}(d) = r\mathrm{HM}(c)$. It follows from (5) that $r\mathrm{HM}(a) = \mathrm{HM}(b)$, contradicting (4).

   In the case $\deg(\mathrm{HM}(c)) = \deg(\mathrm{HM}(d))$ this means that there exists a polynomial $r \in \mathrm{Quot}(A)$ such that $\mathrm{HM}(d) = r\mathrm{HM}(c)$. It follows from (5) that $\deg(\mathrm{HM}(a)) = \deg(\mathrm{HM}(b))$ and $r\mathrm{HM}(a) = \mathrm{HM}(b)$ hold, contradicting (4).

   We conclude that $\mathrm{cnc}(c, d)$ is true. In the same way we find that $\mathrm{cnc}(d, c)$ is true. □

Denote by D the matrix

$$D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 9 *If $M \in \mathrm{SL}_2(C)$ satisfies condition S, then DM and MD satisfy condition S as well.*

Proof. Clear. □

Corollary 10 *Let $y \in C$. If $M \in \mathrm{SL}_2(C)$ satisfies condition S, then $T_{12}(y)M$ satisfies condition S as well.*

Proof. Since $T_{12}(y) = DT_{21}(-y)D^{-1}$, this follows from the previous two lemmas. □

We can now complete the proof of Theorem 7. It suffices to show that $\mathrm{SL}_2(C)$ can not be generated by transvections $T_{12}(y)$, $T_{21}(y)$, $y \in C$.

By assumption there exists a non-zero non-unit $u \in A$. Let M be the matrix

$$M = \begin{pmatrix} 1 + uz^{-1} & u^2 \\ -z^{-2} & 1 - uz^{-1} \end{pmatrix}$$

This matrix M satisfies condition S and therefore no multiplication by elementary matrices can reduce it to the identity matrix. $\square$

Remark.

> Tolhuizen, Hollmann, and Kalker use a similar approach in [10] to show the incompleteness in the case of integer coefficients $A = \mathbf{Z}$ and polynomial rings $A = k[y]$, k a field. These two counter examples are also contained in the seminal paper by Cohn [3].

5 Semiperfect Coefficient Rings

The preceding section showed some limits of lifting steps in the case of integral domains. In this section we give some positive results, which show that a complete factorization into lifting steps is possible if the coefficient ring is given for example by a residue class ring $A = \mathbf{Z}/N\mathbf{Z}$.

An ideal in which every element is nilpotent is called nil.

Theorem 11 *Let A be a commutative local ring and suppose that the unique maximal ideal m in A is nil. Then $B = A[z, z^{-1}]$ and $C = A[z^{-1}]$ are both GE_2-rings.*

Proof. We have to show that a matrix $M \in \mathrm{GL}_2(B)$ can be reduced to the identity matrix by multiplication with elementary transvections and diagonal matrices. By reducing all entries in M modulo the ideal mB, we obtain a matrix $\overline{M} \in \mathrm{GL}_2(B/mB)$. Since $B/mB \cong (A/m)[z, z^{-1}]$ is a Laurent polynomial ring over the residue class field A/m, hence a euclidean ring, we can express $\overline{M}$ as a product of elementary matrices in $\mathrm{GE}_2(B/mB)$, namely $\overline{M} = \overline{E_1}\,\overline{E_2} \cdots \overline{E_r}$. We can lift the matrices $\overline{E_i} \in \mathrm{GE}_2(B/mB)$ to matrices $E_i \in \mathrm{GE}_2(B)$. Thus, the matrix

$$N = ME_r^{-1} \cdots E_2^{-1}E_1^{-1}$$

is of the form

$$N \in \begin{pmatrix} 1+mB & mB \\ mB & 1+mB \end{pmatrix}.$$

Since m is a nil ideal in A, it follows that mB is a nil ideal in B. Consequently, the diagonal entries of N are units in B. By multiplication with two elementary transvections we can reduce N to an invertible diagonal matrix, which proves that B is a GE_2-ring. The reasoning for C is similar. $\square$

The Jacobson radical $J(A)$ of a ring A is the intersection of all maximal left ideals. A ring A is called semilocal if and only if $A/J(A)$ is semisimple. A ring A is said to be semiperfect if and only if A is semilocal and its Jacobson radical $J(A)$ is idempotent lifting. The structure theory of semiperfect rings used in the proof of the following theorem is explained for example in Lam [6].

Theorem 12 *Let A be a commutative semilocal ring and assume that the Jacobson radical $J(A)$ of A is nil. Then $B = A[z, z^{-1}]$ and $C = A[z^{-1}]$ are both* GE_2*-rings.*

Proof. Since the Jacobson radical is nil, it is in particular idempotent lifting. In other words, A is a semiperfect ring. A commutative semiperfect ring is a finite direct product of local rings. Namely, there exists a complete set of primitive orthogonal idempotents e_i such that $A = e_1 A \oplus \cdots \oplus e_r A$, and each $e_i A$ is a local ring. The Jacobson radical of $e_i A$ is given by $e_i J(A)$, and thus is a nil ideal. Therefore, A is isomorphic to a finite direct product of local rings L_i, each with nil maximal ideal. Consequently, B and C are isomorphic to the finite direct products

$$B \cong \prod_{i=1}^{r} L_i[z, z^{-1}] \qquad \text{and} \qquad C \cong \prod_{i=1}^{r} L_i[z^{-1}].$$

It follows from Theorem 11 that the components $L_i[z, z^{-1}]$ and $L_i[z^{-1}]$ are GE_2-rings, therefore B and C are GE_2-rings as well. $\square$

Remark.

> Note that Theorem 11 and Theorem 12 can be generalized to show that B and C are GE_n-rings for any positive integer n, that is, for local or semilocal coefficient rings with nil Jacobson radical an n-channel perfect reconstructing filter bank can be realized with ladder steps (or lifting filters).

6 Conclusion

We studied two-channel filter banks for signals with coefficients in a commutative ring. A straightforward extension is to consider filter banks with more channels. However, in some sense the two-channel case is the most difficult. For example, if A is a euclidean ring that contains a non-unit, then we showed that $A[z^{-1}]$ is not a GE_2-ring. In other words, in this case there exist realizable perfect reconstructing filter banks (in the sense of section 4) that can not be implemented with causal ladder steps. This kind of obstacle can not occur for filter banks with at least three channels, since a theorem by Suslin [9] shows that $A[z^{-1}]$ is a GE_n-ring for $n \geq 3$. For a thorough study of such phenomena the reader should refer to any standard text on Algebraic K-Theory, e. g. [1].

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