On Multirate Filter Bank Structures

Andreas Klappenecker

Department of Mathematics, Texas A&M University
College Station, TX, 77843-3368, USA
andreask@math.tamu.edu

Abstract

We introduce a framework for treating multirate filter banks for signal and image processing applications, as well as communication applications, in a unified way. We show how methods from group and ring theory can be applied to derive structural properties of filter banks.

§1 Introduction

Multirate filter banks are often used in signal and image processing applications to realize a multiresolution decomposition of the input signal. Suitably generalized, these filter banks are also used in communication applications, namely for error correction. Here a convolutional code encoder can be interpreted as a multirate filter bank. We introduce a framework which deals with all these applications in a unified way.

We view the coefficients of the input signal as ring elements. This allows us to deal with the *finite rings* used in communication applications and with *real or complex numbers*, as used in signal or image processing applications. The input signals are indexed by group elements, so that we can treat one dimensional or higher dimensional signals alike.

In more mathematical terms, we will interpret the signals and filters as elements of a group ring. We give an elementary introduction to multirate filter banks for such signals. The algebraic language will be explained thorougly with the help of some familiar examples. One benefit of this approach is that some seemingly complicated constructions, such as multifilter banks for multidimensional signals, can be explained without any hassle.

A second topic will be the decomposition of multirate filter banks into some elementary units, such as ladder structures or lifting steps. The purpose of such a decomposition is twofold: the filter bank can be better understood in these simpler terms, and the decomposition often leads to a faster implementation. We explain a number of factorization results for filter banks in some simple ring theoretic terms.

§2 Two-Channel Filter Banks

We will discuss several different filter bank types below. For simplicity we will always assume that all signals and filters are represented by a finite number of non-zero coefficients. We start with a classical example, namely with two-channel filter banks for one-dimensional real-valued signal sequences. Such a filter bank is illustrated in Figure 1.

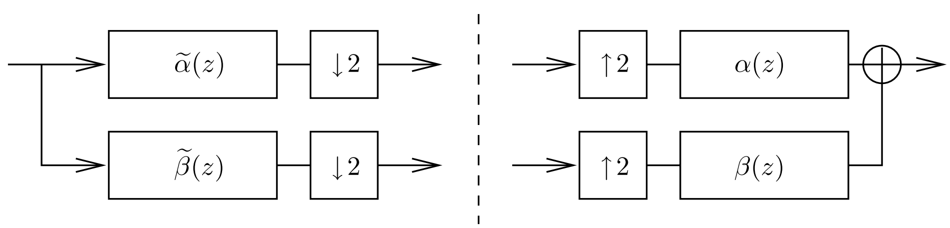

Figure 1: Two-channel filter bank. The analysis (synthesis) filter bank is shown on the left (right) of the dashed line.

It will be convenient to use the z-transform representation for all signals and filters. Thus each signal or filter can be viewed as an element of the Laurent polynomial ring $\mathbf{R}[z, z^{-1}]$. The main advantage of this representation is that a convolution is translated into the usual polynomial product in $\mathbf{R}[z, z^{-1}]$.

In a nutshell, the analysis filter bank works as follows: the input signal is filtered with two analysis filters, and then the sampling rate is reduced by dropping every second coefficient. More precisely, the downsampling operation $[\downarrow 2]$ on $\mathbf{R}[z, z^{-1}]$ is defined by $[\downarrow 2]\, a(z) = a_e(z)$, where $a(z) = a_e(z^2) + z a_o(z^2)$. Thus, the processing of the input signal $s(x)$ yields $d_\alpha(z) := [\downarrow 2]\, \widetilde{\alpha}(z)s(z)$ and $d_\beta(z) := [\downarrow 2]\, \widetilde{\beta}(z)s(z)$ as output sequences.

The *synthesis filter bank* takes two signals $d_\alpha(z)$ and $d_\beta(z)$ as input, applies an upsampling operation, filters these signals with synthesis filters $\alpha(z)$ and $\beta(z)$ respectively, and adds the resulting two signals. This step yields the output signal

$$\hat{s}(z) := \alpha(z)d_\alpha(z^2) + \beta(z)d_\beta(z^2).$$

The z-transform representation led us to consider the ring of Laurent polynomials $\mathbf{R}[z, z^{-1}]$ as an ambient space containing our signals and filters. One particular feature of this ring is that the multiplication of elements is induced by the multiplication of the delay and advance elements, namely $z^m z^n = z^{n+m}$, and the multiplication and addition operations of the real numbers.

Stressing this point of view further, we note that the set of advance and delay elements $G = \{z^k \mid k \in \mathbf{Z}\}$ forms a group under multiplication, which is isomorphic to the additive group of integers $(\mathbf{Z}, +)$. In other words, our ambient space can be viewed as the group ring $\mathbf{R}[G]$, that is, each element of $R[G]$ can be written uniquely in the form

$$\sum_{g\in G} s_g\, g, \qquad s_g \in \mathbf{R},$$

where only finitely many s_g are non-zero.

Of course, all this is rather obvious and the reader might wonder what kind of benefit can be expected from changing some notation. One advantage is that many other filter bank types can be modelled by taking a group ring as an ambient space for signals and filters.

§3 Cyclic Filter Banks

Suppose that the input signals are known to have a fixed maximal length. For example, in an image processing application this might be a row length of n pixels. The filter bank decomposition as in Figure 1, would lead to the need to store an increased number of coefficients. A simple way to avoid this problem is to use cyclic convolutions.

In this case we model the ambient space of signals and filters by the ring of truncated (Laurent) polynomials

$$\mathbf{R}[z, z^{-1}]/\langle 1 - z^{-n}\rangle \cong \mathbf{R}[z^{-1}]/\langle 1 - z^{-n}\rangle.$$

The set of delay elements $G = \{z^{-k \bmod n} \mid k = 0, \ldots, n-1\}$ now constitutes a multiplicative cyclic group of order n (with exponents taken modulo n). The ambient space can again be viewed as a group ring $\mathbf{R}[G] \cong \mathbf{R}[z^{-1}]/\langle 1 - z^{-n}\rangle$.

Suppose that n is even. Denote by $H = \langle z^{-2}\rangle$ the subgroup of index two in G. This group is, of course, isomorphic to the cyclic group

$$H' = \{z^{-k \bmod n/2} \mid k = 0, \ldots, n/2 - 1\}.$$

We can define a downsampling operation $[\downarrow 2]$ by dropping the odd indexed coefficients, followed by some reindexing, namely,

$$[\downarrow 2] : \begin{cases} \mathbf{R}[G] & \longrightarrow & \mathbf{R}[H'] \\ s(z) & \longmapsto & s_e(z), \end{cases}$$

where $s(z) = s_e(z^2) + z^{-1}s_o(z^2)$. The upsampling operation $[\uparrow 2]$ is given by the map $\mathbf{R}[H'] \to \mathbf{R}[G]$, $s_e(z) \mapsto s_e(z^2)$. In other words, the upsampling operation $[\uparrow 2]$ maps the generator z^{-1} of H' to the generator z^{-2} of H.

The cyclic filter bank operates pretty much the same way as the filter bank in the preceding section, though the convolution operations are now replaced by cyclic convolutions (polynomial multiplications modulo $1 - z^{-n}$). Instead of detailing all these steps right now, we use the cyclic and non-cyclic two-channel filter banks as running examples, to illustrate what is going on.

§4 Filter Banks over Group Rings

We assumed in the preceding two sections that the signals and filters are real valued. In some applications it is of interest to allow complex-valued signals or to consider only integer-valued sequences. Finite fields or finite rings are often used in communication applications. In our approach the signal may take values in a ring A. We will always assume that this ring is associative and has an identity element $1 \neq 0$. However, we do not assume that A is commutative. This allows us to deal with multifilters that are matrix-valued.

The unifying theme of the preceding two sections was that the signals are elements of a group ring. We follow this principle, and consider the elements of the group ring AG as signals or filters, where A is a ring and G is a group. Recall that an element $S \in AG$ is given by a formal linear combination

$$S = \sum_{g \in G} s_g\, g, \qquad s_g \in A,$$

with only finitely many $s_g \neq 0$.

In other words, the group ring AG is a free A-module with basis G. Moreover, the group ring is equipped with a multiplication induced by the product in G. More explicitly, this multiplication is defined by

$$\Big(\sum_{g \in G} s_g\, g\Big) \cdot \Big(\sum_{h \in G} t_h\, h\Big) = \sum_{k \in G} \Big(\sum_{gh=k} s_g t_h\Big) k$$

This multiplication is nothing but a convolution, namely

$$\Big(\sum_{g\in G} s_g\, g\Big)\cdot\Big(\sum_{h\in G} t_h\, h\Big) = \sum_{h\in G}\Big(\sum_{g\in G} s_g t_{g^{-1}h}\Big)h.$$

Remark. Usually the group G is identified with its image in the group ring AG. However, this is rather confusing if G is written additively. We agree to identify the additive group G with the multiplicative group obtained by $g \mapsto z^g$ in such a case.

Examples. 1. Take the integers $\mathbf{Z}$ and the real numbers $\mathbf{R}$. We identify the group ring $\mathbf{R}[\mathbf{Z}]$ with the Laurent polynomial ring $\mathbf{R}[z, z^{-1}]$ (or with $\mathbf{R}[G]$ from §2) according to the previous remark. The infinite cyclic group induces the usual convolution operation.

2. Take the finite cyclic group $(\mathbf{Z}/n\mathbf{Z}, +)$ and the real numbers $\mathbf{R}$. Here we recover the example from §3. The finite cyclic group induces the cyclic convolution operation.

3. If we take the additive group $\mathbf{Z}^2$ and the real numbers, then we obtain the group ring $\mathbf{R}[\mathbf{Z}^2]$. This group ring allows us to model two-dimensional signals. We will write the signals and filters in the form $\sum_{n,m\in\mathbf{Z}} s_{n,m} z^{(n,m)}$.

The bottom line of the preceding discussion is that the group ring comes with a convolution operation (which is simply the multiplication in this ring). We go on to define the down- and upsampling operations.

Basically, the downsampling operation can be divided into two steps. In the first step, certain coefficients are dropped. And in a second step, the remaining coefficients are relabeled.

Let G be a group, and H a normal subgroup of finite index in G. In the first step we drop all coefficients that are not indexed by an element of H, that is, we apply the following map

$$S = \sum_{g\in G} s_g\, g \longmapsto \sum_{g\in H} s_g\, g =: S_e.$$

Let H' be a group isomorphic to the subgroup H of the group G. Denote by $r\colon AH \to AH'$ the induced ring isomorphism. In the second step we apply the relabeling isomorphism r to the sequence S_e. Therefore the downsampling operation is defined by

$$[\downarrow]\colon \begin{cases} AG & \longrightarrow \quad AH', \\ S & \longmapsto \quad r(S_e). \end{cases} \tag{1}$$

The *upsampling operation* $[\uparrow]$ is given by the isomorphism r^{-1}. Thus, decimation followed by upsampling gives the projection operation $S \mapsto S_e$.

Examples. 1. The downsampling and upsampling operations in §2 can be recast as follows. The group G is given by the infinite cyclic group generated by z^{-1}. The normal subgroup H is given by the cyclic subgroup generated by z^{-2}. Thus, in the first step, a sequence $s(z) = s_e(z^2) + z^{-1}s_o(z^2)$ is mapped onto $s_e(z^2)$. The group H is isomorphic to the cyclic group $H' = \langle z^{-1} \rangle$. Hence, the relabeling isomorphism r yields $r(s_e(z^2)) = s_e(z)$.

2. In §3, the ambient space was given by the group ring $\mathbf{R}[\mathbf{Z}/n\mathbf{Z}]$, where n is an even integer. Thus $G = \mathbf{Z}/n\mathbf{Z}$, and $H = 2\mathbf{Z}/n\mathbf{Z}$. Now H is isomorphic to $H' = \mathbf{Z}/m\mathbf{Z}$, where $m = n/2$. The description of the downsampling operation in §3 followed just these two steps.

Let A be a ring. Let G be a group containing a normal subgroup H of finite index in G. Suppose that H' is a group isomorphic to H. We denote by $[\downarrow]$ the downsampling operation defined by (1), and by $[\uparrow]$ the corresponding upsampling operation.

An m-channel filter bank for signals in AG is shown in Figure 2. It consists of m analysis filters $F^i \in AG$, and m synthesis filters $G^i \in AG$. Note that we use superscripts to distinguish the different filters. Since we do not iterate any filters, this should not cause confusion.

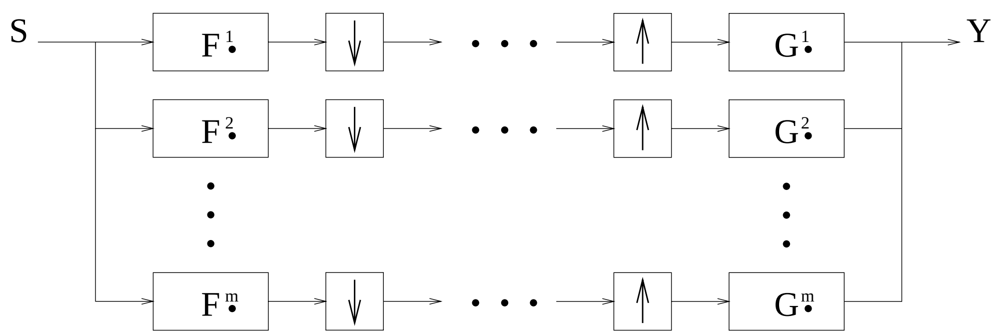

Figure 2: A filter bank for signals in the group ring AG with m channels.

The analysis filter bank works as follows. The input signal S is filtered with the analysis filters F^i. The resulting m intermediate signals $D^i := F^iS$ are then downsampled $E^i := [\downarrow]D^i$.

The synthesis filter bank takes m signals $E^i \in AH'$ as input. These sequences are upsampled $U^i := [\uparrow]E^i$, and subsequently "interpolated" by convolution with the synthesis filters G^i. Finally, the interpolated sequences G^iU^i are summed up, yielding

$$Y = \sum_{i=1}^{m} G^iU^i. \tag{2}$$

§5 Perfect Reconstruction

A filter bank is called *perfect reconstructing* if and only if the output signal Y coincides with the input signal S for all $S \in AG$.

We keep the notations of the previous section. Let us define the τ-component S_τ of a signal $S \in AG$ by

$$S_\tau = \sum_{g \in H} s_{\tau g}\, g,$$

where $\tau \in G$.

Theorem 1 *Let $\mathcal{F}$ be an m-channel filter bank as described in the previous section (and as shown in Figure 1). Denote by T a transversal of H in G. The filter bank $\mathcal{F}$ is perfect reconstructing if and only if*

$$\delta_{\sigma,\tau} = \sum_{i=1}^{m} G^i_\sigma (\tau^{-1} F^i_{\tau^{-1}} \tau)$$

holds for all $\sigma, \tau \in T$.

Proof. Denote by T a left transversal of H in G, that is, G is the disjoint union of the cosets τH, $\tau \in T$. Then $T^{-1} := \{\tau^{-1} | \, \tau \in T\}$ is also a left transversal of H in G.

From the definition of the τ-component, we immediately get

$$S = \sum_{\tau \in T} \tau S_\tau \qquad \text{and} \qquad F^i = \sum_{\sigma \in T^{-1}} \sigma F^i_\sigma .$$

Therefore, the signal $U^i = [\uparrow][\downarrow](F^i S) \in AH$ can be written as

$$U^i = [\uparrow][\downarrow](F^i S) = \sum_{\tau \in T} (\tau^{-1} F^i_{\tau^{-1}})(\tau S_\tau) = \sum_{\tau \in T} \left(\tau^{-1} F^i_{\tau^{-1}} \tau\right) S_\tau .$$

Here we used the fact that $[\uparrow][\downarrow]$ projects on AH, and that the support of $\sigma F_\sigma \tau S_\tau$ is contained in $\sigma \tau H$.

Equation (2) reads now as follows

$$Y = \sum_{i=1}^{m} G^i U^i = \sum_{i=1}^{m} \Big[\Big(\sum_{\sigma \in T} \sigma G^i_\sigma \Big) \sum_{\tau \in T} \left(\tau^{-1} F^i_{\tau^{-1}} \tau S_\tau \right) \Big].$$

Using distributivity and a change of sums, we obtain

$$Y = \sum_{\sigma \in T} \sigma \Big(\sum_{\tau \in T} \Big(\sum_{i=1}^{m} G^i_\sigma \tau^{-1} F^i_{\tau^{-1}} \tau \Big) S_\tau \Big). \tag{3}$$

Hence the polyphase components Y_σ with respect to T are given by

$$Y_\sigma = \sum_{\tau \in T} \Big(\sum_{i=1}^{m} G^i_\sigma \tau^{-1} F^i_{\tau^{-1}} \tau \Big) S_\tau. \tag{4}$$

The signal S coincides with Y iff the polyphase components S_τ and Y_τ coincide for all $\tau \in T$. Thus, perfect reconstruction of all signals $S \in AG$ is ensured if and only if

$$\sum_{i=1}^{m} G^i_\sigma (\tau^{-1} F^i_{\tau^{-1}} \tau) = \delta_{\sigma,\tau} \tag{5}$$

holds, where $\delta_{\sigma,\tau}$ denotes the Kronecker delta function. $\square$

Denote by I the interval $[1, m]$ in the natural numbers. Define the matrices $\mathcal{H}_{tp}$ and $\mathcal{G}_p$ by

$$\mathcal{H}_{tp} = \Big(\tau^{-1} F^i_{\tau^{-1}} \tau \Big)_{(i,\tau)\in I\times T}, \qquad \mathcal{G}_p = \Big(G^i_\tau \Big)_{(i,\tau)\in I\times T}.$$

Applying the isomorphism r to the entries of $\mathcal{H}_{tp}$ and $\mathcal{G}_p$ yields the matrices $\mathcal{H}'_{tp}$ and $\mathcal{G}'_p$. The perfect reconstruction condition (5) can be expressed by the equation $\mathcal{G}'^t_p \mathcal{H}'_{tp} = I$. The preceeding proof suggests the implementation shown in Figure 3.

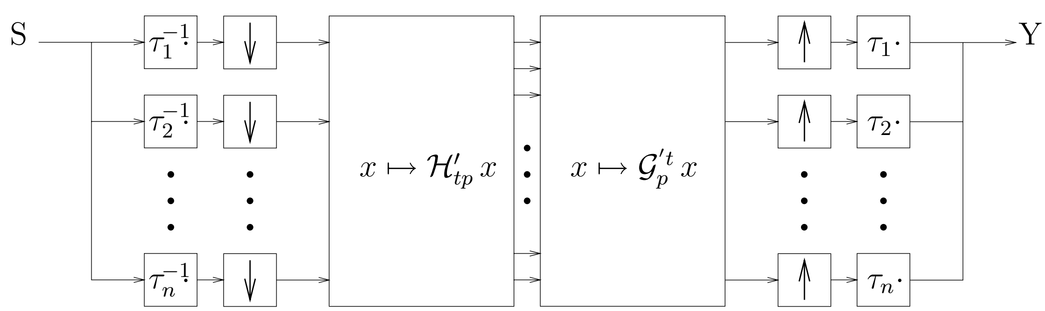

Figure 3: Polyphase implementation of a filter bank. The transpose of $\mathcal{G}'_p$ is the left inverse of $\mathcal{H}'_{tp}$ in a perfect reconstructing filter bank.

Note that the number of channels $m = |\, I \,|$ need not be the same as the index of H in G (which measures the change of the sampling rate). A filter bank is called critically decimated if $|\, I \,| = |\, T \,|$. The examples given in §§2–3 satisfy this property. One application of such filter banks is subband coding. The next two examples show that adding some redundancy by taking $|\, I \,| > |\, T \,|$ can be a desired feature of a filter bank.

§6 Convolutional Encoders

Let $n \leq m$ be positive integers. A convolutional code $\mathcal{C}$ of rate n/m over $\mathbf{F}_2$ is defined to be the image of the homomorphism

$$\begin{array}{ccc} \mathbf{F}_2[z,z^{-1}]^n & \longrightarrow & \mathbf{F}_2[z,z^{-1}]^m \\ U & \longmapsto & \mathcal{H}'_{tp}U, \end{array}$$

where the matrix $\mathcal{H}'_{tp} \in \operatorname{Mat}_{m,n}(\mathbf{F}_2[z,z^{-1}])$ is required to have a left inverse:

$$\mathcal{G}'^{\,t}_p \mathcal{H}'_{tp} = I, \qquad\qquad \mathcal{G}'^{\,t}_p \in \operatorname{Mat}_{n,m}(\mathbf{F}_2[z,z^{-1}]).$$

The matrix $\mathcal{H}'_{tp}$ is called a generator matrix of $\mathcal{C}$.

Let A be the binary field $\mathbf{F}_2$. Let G be the infinite cyclic group generated by z. Denote by H the subgroup of G generated by z^n. This is again an infinite cyclic group, isomorphic to $H' = \langle z \rangle$. The induced ring isomorphism r from AH to AH' satisfies $r(z^n) = z$. Thus, defining the downsampling operation as in (1), we get

$$[\downarrow]\left(\sum_{k\in\mathbf{Z}} s_k z^{-k}\right) = \sum_{k\in\mathbf{Z}} s_{nk} z^{-k}.$$

An information sequence $S \in AG = \mathbf{F}_2[z,z^{-1}]$ can be represented by its τ-components $S_\tau \in AH$, where $\tau \in T = \{1, z, \ldots, z^{n-1}\}$. Equivalently, we may represent S by the vector

$$(r(S_1), r(S_z), \ldots, r(S_{z^{n-1}}))^t \in A[H']^n = \mathbf{F}_2[z,z^{-1}]^n.$$

This vector can be used as input of a convolutional encoder of $\mathcal{C}$. In other words, an encoder of $\mathcal{C}$ is given by the analysis filter bank shown on the left hand side in Figure 3.

Some remarks are in order. In error control coding, the ambient space of information sequences is traditionally chosen to be the field $\mathbf{F}_2((z))$ of Laurent series [4], which contains $AG = \mathbf{F}[z,z^{-1}]$ as a subring. The generator matrix is usually restricted to have entries in $\mathbf{F}_2(z)$, or some subring of it. The definition of the convolutional code corresponds to our definition of a perfect reconstruction filter bank. (I avoided 'rational functions' in the general filter bank theory, since localization of group rings leads to some delicate ring theoretic questions. The case of commutative group rings is of course without problems.)

The encoded sequence is transmitted over a noisy channel. The decoder attempts to reconstruct the original signal. The reconstruction filter bank (shown on the right hand side of Figure 3) is no longer appropriate for this purpose, since a single error could distort several bits in the output sequence. There are numerous decoding procedures available, for example the maximum likelihood decoding procedure by Viterbi, which are more appropriate. We refer the reader to [4, 8, 9] for more details.

However, the reconstruction filter bank can be used to "look inside" the received sequence before the decoding. This feature was used in a convolutional code constructed by Massey and Costello for the Pioneer 10 and Pioneer 11 mission [3].

§7 The Golay Code

The connection between convolutional codes and filter banks is rather obvious. It is perhaps more surprising that certain block codes have a natural filter bank interpretation. We illustrate this for the extended binary Golay code.

The projective line $PL(23)$ is given by the 24 values of the formal ratio x/y, where x, y are elements of the finite field $\mathbf{F}_{23}$. The group $L_2(23) = PSL_2(23)$ acts on this set by linear fractional transforms $z \mapsto (az+b)/(cz+d)$, with $ad - bc = 1$. Denote by Q the set consisting of 0 and the quadratic residues modulo 23:

$$Q = \{0, 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}.$$

Take the orbit of Q under $L_2(23)$, and consider the characteristic functions $PL(23) \rightarrow \mathbf{F}_2$ of the sets in this orbit. The extended binary Golay code $\mathcal{C}_{24}$ is defined to be the linear span of these characteristic functions.

The extended binary Golay code $\mathcal{C}_{24}$ is a $[24, 12, 8]$ code, that is, $\mathcal{C}_{24}$ is a 12-dimensional subspace of $\mathbf{F}_2^{24}$, and the Hamming distance between two different vectors in $\mathcal{C}_{24}$ is at least 8. This remarkable code has been studied in detail, see [2, 7] and the references therein.

The Golay code is a quasi-cyclic code, meaning that a cyclic shift by two of a codeword is again an element of $\mathcal{C}_{24}$. Thus, the Golay code can also be viewed as generated by a cyclic filter bank. We give two essentially different filter banks that generate the Golay code.

Version 1. Let $A = \mathbf{F}_2$. Let G, H, H' all be given by the cyclic group $\mathbf{Z}/12\mathbf{Z}$. In other words, there is no downsampling. We use a two-channel filter bank to map an information vector from AG onto two sequences in AH'. The polyphase matrix $\mathcal{H}'_{tp}$ is given by

$$\mathcal{H}'_{tp} = \begin{pmatrix} 1 \\ 1 + z^{-1} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-8} \end{pmatrix}.$$

Therefore, the output of the first channel is just the information sequence (the encoder is systematic). In fact, this is the filter bank version of the extended quadratic residue construction of $\mathcal{C}_{24}$, see [7, Chap. 16, §6].

Version 2. Let $A = \mathbf{F}_2$. Let $G = \mathbf{Z}/12\mathbf{Z}$, $H = 4\mathbf{Z}/12\mathbf{Z}$, and $H' = \mathbf{Z}/3\mathbf{Z}$. The sampling rate is reduced by four in this version. This leads to an eight-channel filter bank. The polyphase matrix $\mathcal{H}'_{tp}$ is given by

$$\mathcal{H}'_{tp} = \begin{pmatrix} 1+z^{-1} & 0 & z^{-1} & 0 \\ 0 & 1+z^{-1} & z^{-1} & z^{-1} \\ 1 & 1 & 1+z^{-1} & 0 \\ 0 & 1 & 0 & 1+z^{-1} \\ 1+z^{-1} & z^{-1} & 0 & z^{-1} \\ 1 & 1+z^{-1} & z^{-1} & z^{-1} \\ 1 & 1 & 1+z^{-1} & z^{-1} \\ 1 & 0 & 1 & 1+z^{-1} \end{pmatrix}.$$

This version is more attractive from a decoding point of view. The reason is that only four memory cells are needed to realize $\mathcal{H}'_{tp}$. Therefore, the trellis of this block code has only 16 different states. We refer the interested reader to the book of Johannesson and Zigangirov [4] for more information on decoding algorithms. More information on tailbaiting trellises for $\mathcal{C}_{24}$ can be found in [1].

§8 Some Results on Ladder Steps

We give now a simple construction for the perfect reconstructing filter banks discussed in the preceding sections. This construction uses some simple building blocks, which are known in the engineering literature as ladder steps, and in the wavelet literature as lifting steps. Basically it is a factorization of the polyphase matrix into elementary transvections and invertible diagonal matrices. We focus only on the critically sampled case in this section.

Let R be a ring. An elementary transvection in $\mathrm{GL}_n(R)$ is a matrix that differs from the identity matrix in at most one off-diagonal entry. We denote by $\mathrm{GE}_n(R)$ the subgroup of $\mathrm{GL}_n(R)$ generated by elementary transvections and invertible diagonal matrices. The ring R is called *generalized euclidean* if and only if $\mathrm{GE}_n(R) = \mathrm{GL}_n(R)$ for all n.

Thus if the group ring AH is generalized euclidean, then the polyphase matrices $\mathcal{H}_{tp}$ and $\mathcal{G}_p^t$ can be written as a product of elementary transvections and one diagonal matrix. This corresponds to a network of ladder steps (a.k.a. lifting steps).

Theorem 2 *Let F be a field, and let H be the infinite cyclic group. Then $FH = F[z, z^{-1}]$ is a generalized euclidean ring.*

Proof. Let M be an element in $\mathrm{GL}_n(FH)$. The ring $F[z, z^{-1}]$ is euclidean, hence a principal ideal ring. Therefore we can bring M to an invertible diagonal matrix by applying a finite number of row and column operations [6, Chap. III, §7]. Note that row and column operations correspond to multiplications by elementary transvections from the left and right, respectively. Thus $M \in \mathrm{GE}_n(FH)$. □

Theorem 3 *Let F be a field, and let H be a finite cyclic group. Then FH is a generalized euclidean ring.*

Proof. Denote the Jacobson radical of FH by J (thus J is the intersection of all maximal ideals in FH). The group ring FH is a commutative artinian ring. Therefore, FH/J is isomorphic to a finite direct product of fields. It immediately follows that FH/J is a generalized euclidean ring.

Take an arbitrary matrix $M \in \mathrm{GL}_n(FH)$. We can find matrices $S, T \in \mathrm{GE}_n(FH)$ such that the product SMT maps to an invertible diagonal matrix in $\mathrm{GL}_n(FH/J)$. Then the entries on the diagonal of SMT are units as well, since units lift through the Jacobson radical. It follows that $SMT \in \mathrm{GE}_n(FH)$, hence $M \in \mathrm{GE}_n(FH)$. □

§9 Conclusion

The approach to filter bank theory in this paper is clearly biased towards algebraic methods. I wanted to point out the strong structural similarities between rather different applications. I have chosen the cyclic groups as a running example to keep things simple. The filter banks in §2 are well-studied and have numerous applications, see for example [10, 11, 13], and the references therein. The cyclic filter banks in §3 are investigated for example in [12], but from a different point of view. Although the connection between filter banks and convolutional codes is obvious, there seem to exist only a few publications devoted to this subject, see for example [14]. More information on filter banks over group rings can be found in [5].

Acknowledgements. I wish to thank Professor Bob Blakley for valuable comments that improved the presentation of this paper.

References

[1] A. R. Calderbank, G. D. Forney, and A. Vardy. Classification of certain tail-biting generators for the binary golay code. In A. Vardy, editor, *Codes, Curves, and Signals – Common Threads in Communications*, pages 127–153. Kluwer Academic Publishers, Boston, 1998.

[2] J. H. Conway and N. J. A. Sloane. *Sphere Packings, Lattices and Groups*. Springer Verlag, New York, 3 edition, 1999.

[3] D. J. Costello, J. Hagenauer, H. Imai, and S. B. Wicker. Applications of error-control coding. *IEEE Trans. Inform. Theory*, 44(6):2531–2560, 1998.

[4] R. Johannesson and K. Sh. Zigangirov. *Fundamentals of Convolutional Coding*. IEEE Series on Digital and Mobile Communication. IEEE Press, Piscataway, NJ, 1999.

[5] A. Klappenecker. *Algebraische Wavelets*. Forschung und Wissen. GCA-Verlag, Herdecke, 1998. (Dissertation, Universität Karlsruhe).

[6] S. Lang. *Algebra*. Addison-Wesley, 3rd edition, 1993.

[7] F. J. MacWilliams and N. J. A. Sloane. *The Theory of Error Correcting Codes*. Elsevier Science B. V., Amsterdam, 1977.

[8] R. J. McEliece. The algebraic theory of convolutional codes. In V. Pless and W. C. Huffman, editors, *Handbook of Coding Theory*, volume 1. Elsevier Science, New York, 1998.

[9] Ph. Piret. *Convolutional Codes - An Algebraic Approach*. The MIT Press, Cambridge, MA, 1988.

[10] G. Strang and T. Nguyen. *Wavelets and Filter Banks*. Wellesley-Cambridge Press, Wellesley, 1996.

[11] P. P. Vaidyanathan. *Multirate Systems and Filter Banks.* Prentice Hall, 1993.

[12] P. P. Vaidyanathan and A. Kirac. Theory of cyclic filter banks. In *IEEE Int. Conf. on Accoustics, Speech and Signal Processing*, pages 2449–2452. IEEE, 1997.

[13] M. Vetterli and J. Kovačević. *Wavelets and Subband Coding.* Prentice Hall, 1995.

[14] X.-G. Xia. Multirate filterbank approach for error correction codes with applications in partial reponse channels. Available from the author[1].

[1]e-mail: xxia@ee.udel.edu