Construction of Algebraic Wavelet Coefficients

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Summary

In this paper we discuss a method for construction of algebraic wavelet coefficients, i.e., wavelet coefficients lying in an algebraic extension field of $\mathbb{Q}$. The method relies on a strengthened version of a theorem due to L. FEJÉR and F. RIESZ. As an application, we prove that the Daubechies wavelets have algebraic wavelet coefficients. We show that there exist uncountably many transcendent scaling coefficient sequences. Furthermore, we prove that the set of parameters for algebraic wavelet coefficient sequences (up to a given length) is dense in the parameter space of the Pollen parametrization. Algebraic wavelet coefficient sequences may lead to faster processing units in VLSI implementations of the fast wavelet transform.

1 INTRODUCTION

First, we introduce the notion of algebraic wavelet coefficients, i.e., wavelet coefficients lying in an algebraic extension field of $\mathbb{Q}$. For the sake of simplicity we restrict our attention to compactly supported orthonormal wavelets in one dimension (for the basic notions of wavelet analysis the reader is referred to DAUBECHIES [1], or MEYER [2]).

Recall that an orthonormal compactly supported scaling function $\varphi(x) \in L^2(\mathbb{R})$ satisfies a dilation equation

$$\varphi(x) = \sum_{n \in \mathbb{Z}} h_n \sqrt{2}\, \varphi(2x-n), \quad \text{with} \quad h_n \in \mathbb{C}. \tag{1}$$

The coefficients h_n are called scaling coefficients. Note that for compactly supported scaling functions only a finite number of non-zero coefficients h_n is involved. Out of a given scaling function we can easily construct a wavelet ψ through

$$\psi(x) = \sum_{n \in \mathbb{Z}} g_n \sqrt{2}\, \varphi(2x-n), \quad \text{with} \quad g_n \in \mathbb{C},$$

where the wavelet coefficients g_n may be obtained from the scaling coefficients in the following way

$$g_n = (-1)^n\, h_{1-n}.$$

Definition 1 *We call a sequence (h_n) of scaling coefficients algebraic, if every coefficient h_n is algebraic over $\mathbb{Q}$, otherwise we call this sequence transcendent. Algebraic or transcendent wavelet coefficient sequences are defined analogously.*

We denote by $\mathbb{Q}^a$ the algebraic closure of $\mathbb{Q}$. The field $\mathbb{Q}^a$ is interpreted as a subfield of $\mathbb{C}$.

By slight abuse of language we call a wavelet (resp. scaling function) algebraic if its wavelet coefficient sequence (resp. scaling coefficient sequence) is algebraic.

Remark 2 *Clearly, a wavelet coefficient sequence (g_n) is algebraic iff the corresponding scaling coefficient sequence (h_n) is algebraic.*

Hence it is enough to study the construction of scaling coefficient sequences.

2 CONSTRUCTION

The construction of wavelets in one dimension is now well understood. Consider the following trigonometric polynomial m_0 associated with a compactly supported scaling function φ :

$$m_0(\omega) := \frac{1}{\sqrt{2}} \sum_n h_n\, e^{-in\omega}. \tag{2}$$

With the help of this trigonometric polynomial we can rewrite the dilation equation (1) in Fourier space as follows:

$$\hat{\varphi}(\omega) = m_0(\omega/2)\, \hat{\varphi}(\omega/2).$$

The trigonometric polynomial $m_0(\omega)$ plays a crucial role in the construction of scaling functions.

From the orthonormality of the $\mathbb{Z}$-translated scaling functions $\varphi(x-n)$, with $n \in \mathbb{Z}$, we deduce the following simple relation for the "squared version" $M(\omega) := |\, m_0(\omega)\,|^2$ of the trigonometric polynomial $m_0(\omega)$:

$$M(\omega) + M(\omega+\pi) = |\, m_0(\omega)\,|^2 + |\, m_0(\omega+\pi)\,|^2 = 1.$$

This relationship for $M(\omega)$ is easy to satisfy. Moreover, all possible functions $M(\omega)$ are described in the book of DAUBECHIES in Proposition 6.1.2 [1, p. 171].

We may make use of our full knowledge about the functions $M(\omega)$ in the construction of scaling functions. In a first step we choose such a function $M(\omega)$.

The following theorem assures that we can extract a "square root" $m_0(\omega)$ out of $M(\omega)$. The trigonometric polynomial $m_0(\omega)$ characterizes a scaling function, which in turn can be defined in Fourier space via an infinite product

$$\hat{\varphi}(\omega) := (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\omega).$$

In order to assure the convergence of this product, we necessarily have $m_0(0) = 1$.

If we start from a trigonometric polynomial $M(\omega)$ with *algebraic* coefficients, the resulting trigonometric polynomial $m_0(\omega)$ has algebraic coefficients too. Taking an appropriate $M(\omega)$ yields a bona fide scaling function φ with algebraic scaling coefficients.

In signal processing applications it is often desirable to use wavelets with real valued coefficients. The following theorem shows that we are able to construct real algebraic wavelets. The proof of this theorem is constructive.

Theorem 1 (Fejér-Riesz)

1. *Let $A(\omega)$ be a real nonnegative trigonometric polynomial which is invariant under $\omega \mapsto -\omega$. Obviously, $A(\omega)$ may be written in the following form:*

$$A(\omega) = \sum_{m=0}^{M} a_m \cos m\,\omega, \quad \textit{with} \quad a_m \in \mathbb{R}.$$

Then it is possible to construct a real trigonometric polynomial $B(\omega)$ of the same order M, where $B(\omega)$ is of the form

$$B(\omega) = \sum_{m=0}^{M} b_m\, e^{im\omega}, \quad \textit{with} \quad b_m \in \mathbb{R},$$

such that $A(\omega) = |\,B(\omega)\,|^2$.

2. *If the coefficients a_m of the trigonometric polynomial $A(\omega)$ are real algebraic, that is, $a_m \in \mathbb{Q}^a \cap \mathbb{R}$, then the coefficients b_m of $B(\omega)$ are also real algebraic.*

The first part of this theorem dates back to 1915. L. FEJÉR conjectured the first part of this theorem and was able to prove it for some special cases. F. RIESZ gave a general proof of this conjecture in the same year (cf. [3], see also [4]).

The second part of this theorem is our algebraic reformulation of this theorem. The proof is given in appendix A.

As an application, we briefly sketch the construction of the famous Daubechies wavelets (cf. DAUBECHIES [5]). For the construction of the Daubechies wavelet of order N we start with the trigonometric polynomial

$$M_N(\omega) = \left(\cos^2(\omega/2)\right)^N \; P_N\left(\sin^2(\omega/2)\right),$$

where

$$P_N(x) = \sum_{k=0}^{N-1} \binom{N-1+k}{k} x^k.$$

We can express $M_N(\omega)$ as a trigonometric polynomial in $\cos\omega$ with rational coefficients. Applying the Fejér-Riesz theorem, we extract $m_{0,N}(\omega)$ from $M_N(\omega) = |\,m_{0,N}(\omega)\,|^2$. The corresponding scaling function is determined by the infinite product

$$\hat{\varphi}_N(\omega) := (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_{0,N}(\omega).$$

As a result we immediately get the following important

Corollary 3 *The wavelet coefficients of the Daubechies wavelets are algebraic.*

3 DENSITY

In the preceeding section, we found an infinite family of algebraic wavelets. It is natural to ask if there exist transcendent wavelet coefficient sequences at all. As we will see, there exist uncountably many!

Note, that from now on, we are going to use the convenient normalization $\sum_i h_i = 2$ for scaling coefficient sequences.

According to POLLEN [6] and WELLS [7] all real scaling coefficient sequences of length four or less are given by

$$\begin{aligned}
h_0(\theta) &= \frac{1}{2}\left(1 - \cos\theta + \sin\theta\right), \\
h_1(\theta) &= \frac{1}{2}\left(1 + \cos\theta + \sin\theta\right), \\
h_2(\theta) &= \frac{1}{2}\left(1 + \cos\theta - \sin\theta\right), \\
h_3(\theta) &= \frac{1}{2}\left(1 - \cos\theta - \sin\theta\right),
\end{aligned}$$

with $\theta \in \mathbb{R}$ (up to a trivial shift).

The difference of $h_1(\theta)$ with $h_2(\theta)$ gives

$$\sin\theta = h_1(\theta) - h_2(\theta).$$

As the range of the sine function is the interval $[-1, 1]$, there are uncountably many transcendent values in the range of the sine function. Hence there are uncountably many transcendent scaling coefficient sequences of length four (or less).

Lemma 4 *There exist uncountably many transcendent scaling coeffient sequences.*

In spite of this fact, it is part of the "folklore" among wavelet theorists that many compactly supported wavelets have algebraic wavelet coefficients. To explain this, we prove the following

Theorem 2 *The set of parameters for algebraic scaling coefficients (with prescribed maximal length) is dense in the corresponding parameter space of the Pollen parametrization.*

For a proof see appendix B.

We end this section by an explicit construction of transcendent scaling coefficient sequences. Again, we use the parametrization of all real scaling coefficient sequences of length up to four. It is easy to see that the scaling coefficient sequence $h_0(\theta), \ldots, h_3(\theta)$ is transcendent iff $\sin\theta$ is transcendent. As a consequence of the Hermite-Lindemann theorem, the function $\sin\theta$ is transcendent for all algebraic $\theta \neq 0$ (see e.g. JACOBSON [8, p. 287]). Thus, for algebraic $\theta \neq 0$ the scaling coefficient sequence $h_0(\theta), \ldots, h_3(\theta)$ is transcendent.

With the use of the parametrization of scaling coefficient sequences due to D. POLLEN [6] it is possible to extend this result to arbitrary longer sequences (e.g., by taking an algebraic number not equal to zero for one parameter and rational multiples of π for all other parameters in this parametrization).

For further details the reader is referred to KLAPPENECKER [9, p. 67-73].

4 AN EXAMPLE

What does the construction of algebraic wavelets buy? Let us demonstrate the basic techniques by an example.

In signal processing we often have to deal with sequences over the rationals $\mathbb{Q}$. The scaling coefficients of the Daubechies scaling function of order 2 are given by

$$h_0 = \frac{1+\sqrt{3}}{4}, \quad h_1 = \frac{3+\sqrt{3}}{4},$$
$$h_2 = \frac{3-\sqrt{3}}{4}, \quad h_3 = \frac{1-\sqrt{3}}{4}.$$

For exact calculation with the scaling coefficients h_i we need to perform calculations in a field extension of $\mathbb{Q}$. Typically, the field of real numbers $\mathbb{R}$ is used. In spite of exaggerated arithmetical units we get round off errors by approximating e.g. $\sqrt{3}$. Instead, we may compute in the field $\mathbb{Q}(\sqrt{3})$ by using the following field isomorphism

$$\mathbb{Q}(\sqrt{3}) \cong \mathbb{Q}[x]/\langle x^2-3\rangle \,.$$

All calculations can now be performed with the help of modular polynomial arithmetic with coefficients in $\mathbb{Q}$. The resulting arithmetical units are appealing for VLSI implementations since it is possible to avoid costly floating point units.

5 CONCLUSION

We have introduced the notion of algebraic scaling coefficients. Some properties of algebraic scaling coeffients have been shown. The main techniques were briefly sketched in a tiny example. It is noteworthy that VLSI implementations may result in faster processing units than conventional techniques, even when an exact calculation is not necessary.

A PROOF OF THEOREM 1

To prove the second part of the theorem, we mimic the proof of the first part given in DAUBECHIES [1].

First of all, note that we can write the nonnegative trigonometric polynomial $A(\omega)$ as a polynomial p_A in $\cos\omega$ of same degree over the field $F := \mathbb{Q}(a_0,\dots,a_M)$.

We denote by $\mathrm{lc}(p)$ the leading coefficient of the polynomial p. Furthermore, all fields in this proof are considered to be subfields of $\mathbb{Q}^a \subset \mathbb{C}$.

1. Build the minimal splitting field $E \subset \mathbb{Q}^a$ of $p_A(c)$. This field is an algebraic extension field over F. The polynomial p_A can be factored over E :

$$p_A(c) = \mathrm{lc}(p_A)\prod_{j=1}^{M}(c-c_j).$$

2. By substituting $(z+z^{-1})/2$ for c in $p_A(c)$ and multiplying with z^M, we get a polynomial

$$\begin{aligned} P_A(z) &= \mathrm{lc}(p_A)\, z^M \prod_{j=1}^{M}\left(\frac{z+z^{-1}}{2} - c_j\right) \\ &= \mathrm{lc}(p_A)\prod_{j=1}^{M}\left(\frac{1}{2} - c_j z + \frac{1}{2}z^2\right). \end{aligned}$$

On the unit circle this polynomial coincides with $A(\omega)$, i.e., $A(\omega) = e^{iM\omega}P_A(e^{-i\omega})$.

3. The trigonometric polynomial $B(\omega)$ is constructed with the help of zeros of the polynomial P_A. Before going any further, we need to collect some more information on the zeros of P_A.

The zeros of a factor $(\frac{1}{2} - c_j z + \frac{1}{2}z^2)$ are of the form

$$z_j := c_j + \sqrt{c_j^2-1}, \text{ and } z_j^{-1} := c_j - \sqrt{c_j^2-1}.$$

(a) With c_j not real, we necessarily also have $\overline{c}_j$ as a root of p_A, since the polynomial p_A has real coefficients. Then we have four zeros $z_j, \overline{z}_j, z_j^{-1}$, and $\overline{z}_j^{-1}$ of the polynomial

$$\left(\frac{1}{2} - c_j z + \frac{1}{2}z^2\right)\left(\frac{1}{2} - \overline{c}_j z + \frac{1}{2}z^2\right).$$

(b) If c_j is real and $|\,c_j\,| \geq 1$, then the polynomial $(\frac{1}{2} - c_j z + \frac{1}{2}z^2)$ has two real roots r_j and r_j^{-1}.

(c) If c_j is real and $|\,c_j\,| < 1$, then the zeros of $(\frac{1}{2} - c_j z + \frac{1}{2}z^2)$ are complex conjugate to each other. Furthermore, they have an absolute value of 1. These zeros must have even multiplicity in P_A, otherwise we get a contradiction to the nonnegativity of A. We may also group these zeros of P_A as quadruples $z_j, \overline{z}_j, z_j^{-1}$, and $\overline{z}_j^{-1}$.

4. Denote by E' the minimal splitting field of $P_A(z)$. According to the discussion of the roots in the preceeding step, the polynomial P_A can be factorized over E' into the following special form:

$$P_A(z) = 2^{-M}\,\mathrm{lc}(p_A)\cdot\left[\prod_{l=1}^{K}(z-r_l)(z-r_l^{-1})\right]\cdot$$
$$\cdot\left[\prod_{j=1}^{J}(z-z_j)(z-\overline{z}_j)(z-z_j^{-1})(z-\overline{z}_j^{-1})\right],$$

where $r_j \in \mathbb{R}$ and $K + 2J = M$.

5. The main trick is that on the unit circle we have the following identity for the factors $(z-z_j)(z-\overline{z}_j^{-1})$:

$$|\,(e^{-i\omega}-z_j)(e^{-i\omega}-\overline{z}_j^{-1})\,| = |\,z_j\,|^{-1}\,|\,e^{-i\omega}-z_j\,|^2.$$

6. In a last step, a trigonometric polynomial $B(\omega)$ is constructed out of $A(\omega)$ such that $A(\omega) = |B(\omega)|^2$. With a judicious choice of zeros we can guarantee that the coefficients of $B(\omega)$ are real. Let

$$P_B(z) := \nu \left[\prod_{j=1}^{J} (z - z_j)(z - \overline{z}_j)\right] \cdot \left[\prod_{l=1}^{K} (z - r_l)\right]$$

with a normalization factor

$$\nu := \left[2^{-M} \,|\, \mathrm{lc}(p_A) \,|\, \prod_{j=1}^{J} |\, z_j \,|^{-2} \prod_{k=1}^{K} |\, r_k \,|^{-1}\right]^{1/2}.$$

Then $B(\omega) = P_B(e^{-i\omega})$ is the desired trigonometric polynomial of order M with real algebraic coefficients.

B PROOF OF THEOREM 2

The parametrization of all compactly supported wavelets due to D. POLLEN [6] relies on a bijection between the loop group $SU_I(2, \mathbb{R}[z, 1/z])$ and the set of real scaling coefficients. The elements of this group are 2×2-matrices with finite Laurent polynomials over $\mathbb{R}$ as entries. Moreover, POLLEN derives parametrizations for all scaling coefficient sequences up to a given length (the parametrization we use here is called "*characterization of noncentered $\leq k$ systems*" in [6]).

First we need to introduce some notation. Let $h(z) = \sum_{n \in \mathbb{Z}} h_n z^n$ be a finite Laurent polynomial. We extend the complex conjugation operation of the field $\mathbb{C}$ to a conjugation operation on the ring of Laurent polynomials by defining

$$\widetilde{h(z)} := \sum_{n \in \mathbb{Z}} \overline{h_n}\, z^{-n}.$$

Recall that the group $SU_I(2, \mathbb{R}[z, 1/z])$ is freely generated by a set InvFactors($\mathbb{R}$) (see e.g. [10], [6]). The elements of this set are of the form

$$U_\theta(z) := \begin{pmatrix} u_\theta(z) & v_\theta(z) \\ -\widetilde{v_\theta(z)} & \widetilde{u_\theta(z)} \end{pmatrix}$$

where $u_\theta(z)$ and $v_\theta(z)$ are defined as follows:

$$\begin{aligned} u_\theta(z) &:= \left(\frac{1 - \cos\theta}{2}\right) z^{-1} + \left(\frac{1 + \cos\theta}{2}\right), \\ v_\theta(z) &:= \left(\frac{-\sin\theta}{2}\right) + \left(\frac{\sin\theta}{2}\right) z. \end{aligned}$$

Roughly speaking, all scaling coefficient sequences of length k or less can be obtained by a product of $(k-2)/2$ matrices of the form U_θ or U_θ^{-1}, with $\theta \in [0, 2\pi)$ (again, see POLLEN [6] for details). Note that this parametrization describes scaling coefficient sequences only up to a trivial $2\mathbb{Z}$-translation.

From a topological point of view, the parameter space for length k (or less) scaling coefficient sequences is given by a torus $\mathbb{T}^{(k-2)/2}$. We describe a dense set of parameters in the torus, which correspond to algebraic scaling coefficient sequences.

For rational multiples of π, the coefficients of the polynomials $u_\theta(z)$ and $v_\theta(z)$ are algebraic. The rational multiples of π are dense in the interval $[0, 2\pi)$. From this fact one easily deduces the theorem.

Remark 5 *Note that in this parametrization several parameters may give the same scaling coefficient sequence (modulo a $2\mathbb{Z}$-translation). Of course, we may identify these points by an obvious equivalence relation R. Then the parameter space can be viewed as a pinched torus $\mathbb{T}^{(k-2)/2}/R$. Clearly, the density argument carries over to the set $\mathbb{T}^{(k-2)/2}/R$ equipped with the quotient topology.*

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