Why σ-Algebras are Needed

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In an undergraduate probability course, you probably encountered random variables X that is **uniformly distributed** in $[0, 1]$.

In particular, this means that

$$\begin{aligned}
\Pr[a \leqslant X \leqslant b] &= b - a, \\
\Pr[a < X \leqslant b] &= b - a, \\
\Pr[a \leqslant X < b] &= b - a, \\
\Pr[a < X < b] &= b - a,
\end{aligned}$$

for all real numbers a and b such that $0 < a \leqslant b \leqslant 1$.

Need for σ-Algebras

Proposition

There does not exist a probability measure on the power set of $[0,1]$ that

1. assigns each interval its length and
2. is invariant under shifts $\pmod 1$,

In other words, if we want to have a probability measure with the very reasonable properties (1) and (2), then we need to give up something. The proposition implies that we need to give up that every subset of $(0,1]$ is a measurable set. Fortunately, most sets that occur in practice can be defined as measurable sets using the Borel σ-algebra.

Shifts

For real numbers $x, y \in [0, 1]$, we define

$$x \oplus y = \begin{cases} x + y & \text{if } x + y \leqslant 1, \\ x + y - 1 & \text{if } x + y > 1. \end{cases}$$

For a subset A of $[0, 1]$ and a real number y, we define

$$A \oplus r = \{a \oplus r \mid a \in A\}.$$

A probability measure is invariant under shifts if and only if

$$\Pr[A \oplus r] = \Pr[A]$$

for all $A \subseteq [0, 1]$.

Proof of the Proposition 1/3

We define an equivalence relation $\sim$ on $[0,1]$ by

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbf{Q}.$$

The H be a transversal of $\sim$, so H contains one element of each equivalence class. W.l.o.g. $0 \notin H$. Then the interval $(0,1]$ is contained in

$$\bigcup_{r \in \mathbf{Q}} (A \oplus r).$$

Proof of the Proposition 2/3

Seeking a contradiction, let us assume that a probability measure satisfying (1) and (2) exists.

If x and y distinct rational numbers in $[0, 1)$, then

$$(H \oplus x) \cap (H \oplus y) = \varnothing.$$

By countable additivity of the probability measure, we have

$$\Pr[(0, 1]] = \sum_{r \in \mathbf{Q}, r \in [0,1)} \Pr\left[H \oplus r\right].$$

By property (1), the left-hand side is equal to

$$1 = \Pr[(0,1]]$$

By the shift-invariance property (2), the right-hand side is equal to

$$\sum_{r \in \mathbf{Q}, r \in [0,1)} \Pr[H],$$

so it is either 0 or $+\infty$.

Thus, we get

$$\Pr[(0,1]] \neq \sum_{r \in \mathbf{Q}, r \in [0,1)} \Pr[H \oplus r],$$

contradicting that both sides are equal.

There exist many more results of a similar nature. For instance, on can show the following proposition:

Proposition

There does not exist a probability measure on the power set of $(0, 1]$ *such that*

$$\Pr[(0,1]] = 1$$

and $\Pr[\{x\}] = 0$ *for all* x *in* $(0, 1]$.

Conclusions

We can conclude that the power set of $[0, 1]$ is too big to define reasonable probability measures that are subject to mild constraints. We need to replace the power set by a σ-Algebra.