The Lovász Local Lemma

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The Main Idea of the Probabilistic Method

Suppose that we want to prove the **existence** of a combinatorial object that has certain properties.

In the **probabilistic method**, we approach this problem by defining a sample space of combinatorial objects and showing that a randomly chosen element of this space has the desired properties with positive probability.

One Formulation of the Probabilistic Method

Let $B_1, B_2, \ldots, B_n$ be "bad" events in a probability space that we would like to avoid. Our goal is to show that one can avoid all bad events with nonzero probability,

$$\Pr\left[B_1^c \cap B_2^c \cap \cdots \cap B_n^c\right] > 0.$$

Sieve Methods

Proposition

If $B_1, B_2, \ldots, B_n$ are events, then by the union bound

$$\Pr[B_1 \cup B_2 \cup \cdots \cup B_n] \leqslant \Pr[B_1] + \Pr[B_2] + \cdots + \Pr[B_n].$$

Therefore, if $\Pr[B_1] + \Pr[B_2] + \cdots + \Pr[B_n] < 1$, *then*

$$\Pr\left[B_1^c \cap B_2^c \cap \cdots \cap B_n^c\right] > 0.$$

This result is obvious, since

$$\Pr[B_1 \cup B_2 \cup \cdots \cup B_n] + \Pr[(B_1 \cup B_2 \cup \cdots \cup B_n)^c] = 1.$$

Proposition

If $B_1, B_2, \dots, B_n$ are mutually independent events such that $\Pr[B_k] < 1$ for all k in the range $1 \leqslant k \leqslant n$, then

$$\Pr\left[B_1^c \cap B_2^c \cap \cdots \cap B_n^c\right] > 0.$$

Indeed, if $B_1, B_2, \dots, B_n$ are mutually independent events, then $B_1^c, B_2^c, \dots, B_n^c$. Furthermore, $\Pr[B_k^c] = 1 - \Pr[B_k] > 0$ for all k. Therefore,

$$\Pr\left[B_1^c \cap B_2^c \cap \cdots \cap B_n^c\right] = \Pr[B_1^c]\Pr[B_2^c]\cdots\Pr[B_n^c] > 0.$$

Disadvantages

The independence sieve allows for large probabilities of bad events. The requirement that all bad events are mutually independent is a very strong requirement.

The counting sieve allows for some dependencies among the bad events, but the probabilities of the bad events need to be rather small.

The Lovász local lemma allows for some dependencies and for larger probabilities of bad events.

The Lovász Local Lemma

A Fact About Conditional Probabilities

Proposition

$$\Pr[A \mid B \cap C] = \frac{\Pr[A \cap B \mid C]}{\Pr[B \mid C]}$$

Proof.

$$\begin{aligned}
\frac{\Pr[A \cap B \mid C]}{\Pr[B \mid C]} &= \frac{\Pr[A \cap B \cap C] / \Pr[C]}{\Pr[B \cap C] / \Pr[C]} \\
&= \frac{\Pr[A \cap B \cap C]}{\Pr[B \cap C]} \\
&= \Pr[A \mid B \cap C].
\end{aligned}$$

□

Definition

Let $B_1, B_2, \ldots, B_n$ be events. A graph G on the vertices $\{1, 2, \ldots, n\}$ is called a **dependency graph** for the events $B_1, B_2, \ldots, B_n$ if and only if B_k is mutually independent of all events B_ℓ such that (k, ℓ) is not an edge of G.

The only dependencies that exist are modeled by edges in G.

The Symmetric Lovász Local Lemma

Proposition

Let $B_1, B_2, \ldots, B_n$ be events with dependency graph G such that for each k in the range $1 \leqslant k \leqslant n$, we have

$$\Pr[B_k] \leqslant p \quad \textit{and} \quad \deg(k) \leqslant d,$$

and

$$4dp < 1.$$

Then

$$\Pr[B_1^c \cap B_2^c \cap \cdots \cap B_n^c] > 0.$$

Proof.

We can rewrite $\Pr[B_1^c \cap B_2^c \cap \cdots \cap B_n^c]$ in the form

$$\Pr[B_1^c \cap B_2^c \cap \cdots \cap B_n^c] = \prod_{k=1}^{n} \Pr[B_k^c \mid B_1^c \cap \cdots \cap B_{k-1}^c].$$

It suffices to show that $\Pr[B_k \mid B_1^c \cap \cdots \cap B_{k-1}^c] \leqslant 2p$ for all k. Indeed, this implies

$$\Pr[B_1^c \cap B_2^c \cap \cdots \cap B_n^c] \geqslant \prod_{k=1}^{n} (1-2p) > 0,$$

which proves our claim.

Proof. (Continued)

We prove by induction on s that if $|S| \leqslant s$, then

$$\Pr\left[B_k \,\middle|\, \bigcap_{j \in S} B_j^c\right] \leqslant 2p$$

for all k.

Basis. For $S = \varnothing$, this follows from our hypothesis $\Pr[B_k] \leqslant p$.

Inductive Step. Since our conditions are symmetric, we can without loss of generality renumber our index set such that $k = n$, $S = \{1, 2, \ldots, s\}$ and $(k, x) \notin G$ for all $x > d$. Now

$$\Pr[B_n \mid B_1^c \cap \cdots \cap B_s^c] = \frac{\Pr[B_n \cap B_1^c \cap \cdots \cap B_d^c \mid B_{d+1}^c \cap \cdots \cap B_s^c]}{\Pr[B_1^c \cap \cdots \cap B_d^c \mid B_{d+1}^c \cap \cdots \cap B_s^c]}.$$

Proof. (Continued)

We bound the numerator

$$\begin{aligned}\Pr[B_n \cap B_1^c \cap \cdots \cap B_d^c \mid B_{d+1}^c \cap \cdots \cap B_s^c] &\leqslant \Pr[B_n \mid B_{d+1}^c \cap \cdots \cap B_s^c] \\ &= \Pr[B_n] \leqslant p\end{aligned}$$

as B_n is mutually independent of the events $B_{d+1}, \ldots, B_s$.

We lower-bound the denominator

$$\begin{aligned}\Pr[B_1^c \cap \cdots \cap B_d^c \mid B_{d+1}^c \cap \cdots \cap B_s^c] &\geqslant 1 - \sum_{k=1}^{d} \Pr[B_k \mid B_{d+1}^c \cap \cdots \cap B_s^c] \\ &\geqslant 1 - \sum_{k=1}^{d} 2p \\ &= 1 - 2pd \geqslant 1/2.\end{aligned}$$

Proof. (Continued)

In conclusion, we have

$$\begin{aligned}\Pr[B_n \mid B_1^c \cap \cdots \cap B_s^c] &= \frac{\Pr[B_n \cap B_1^c \cap \cdots \cap B_d^c \mid B_{d+1}^c \cap \cdots \cap B_s^c]}{\Pr[B_1^c \cap \cdots \cap B_d^c \mid B_{d+1}^c \cap \cdots \cap B_s^c]} \\ &\leqslant p/(1/2) = 2p.\end{aligned}$$

It follows by induction that

$$\begin{aligned}\Pr[B_1^c \cap B_2^n \cap \cdots \cap B_n^c] &= \prod_{k=1}^{n} \Pr[B_k^c \mid B_1^c \cap \cdots \cap B_{k-1}^c] \\ &\geqslant \prod_{k=1}^{n} (1-2p) \geqslant 0\end{aligned}$$

as claimed.

Ramsey Numbers

We want to derive a slightly better lower bound on the diagonal Ramsey numbers

$$R(k, k)$$

using the Lovász local lemma.

Proposition

If

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} < 1$$

then $R(k,k) > n$.

Proof.

Consider a random 2-coloring of K_n. If S is a subset of k vertices, then M_S denotes the event that the induces subgraph on S is monochromatic.

Let G be the dependency graph with vertex set the set of all k-subsets of the vertex set of K_n. There is an edge (S, T) between two vertices in G if and only if $|S \cap T| \geqslant 2$.

Then M_S is mutually independent of all M_T such that $(S, T) \notin G$, since M_T gives only information about edges outside of S.

Proof. (Continued)

Each node S of G has degree

$$d = |\{T : |S \cap T| \geqslant 2\}| \leqslant \binom{k}{2}\binom{n}{k-2}.$$

The probability that S is monochromatic is given by

$$p = \Pr[M_S] = 2^{1-\binom{k}{2}}.$$

Proof. (Continued)

Since

$$4pd = \binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} < 1$$

by hypothesis, the Lovász Local Lemma implies that

$$\Pr\left[\bigcap_{S\in\binom{[n]}{k}} M_S^c\right] > 0.$$

Therefore, there must exist a coloring of K_n that does not have any monochromatic k-cliques. So $R(k,k) > n$.

Edge-Disjoint Paths

Disjoint Paths

Problem

Assume that n pairs of users want to simultaneously communicate over a network. Our goal is to find for each pair their own communication path that does not share edges with others.

Let F_i be the set of edges that pair i could use if there were no other users.

Proposition

Let m and k be such that $8nk/m \leqslant 1$. If

1. $|F_i| \geqslant m$ *for all* i *and*
2. *for all* $i \neq j$ *and any path* $P' \in F_i$ *there are at most* k *paths* $P'' \in F_j$ *such that* P' *and* P'' *have at least one common edge,*

then it is possible to choose one path from each set F_i *such that none of the chosen paths have common edges.*

Proof.

It is sufficient to consider the case $|F_i| = m$ for all i. Choose a random path from each F_i. Let $E_{i,j}$ be the event that the paths chosen from F_i and F_j have at least one common edge.

Whatever path P' we choose from F_i, there are m ways of choosing P'' from F_j and at most k of them have a common edge with P'. If we set $p = k/m$, then it follows that

$$\Pr[E_{i,j}] \leqslant k/m = p.$$

Proof. (Continued)

Since $E_{i,j}$ is mutually independent of $\{E_{s,t} \mid \{s,t\} \cap \{i,j\} = \varnothing\}$, the dependency graph has degree $d < 2n$. Hence,

$$4dp < 4(2n)\frac{k}{m} = \frac{8nk}{m} \leqslant 1.$$

It follows from the Lovász Local Lemma that

$$\Pr\left[\bigcap_{\{i,j\}\subseteq[n], i\neq j} E_{i,j}^c\right] > 0,$$

meaning that one can choose a path from each F_i such that all n paths do not share any edges.

Satisfiability

Any k-SAT formula in which no variable appears in too many clauses has a satisfying assignment. More precisely, we have the following proposition.

Proposition

If no variable in a k-SAT formula appears in more than $T = 2^{k-2}/k$ clauses, then the formula has a satisfying assignment.

Proof.

Assign a truth value uniformly at random to each of the variables. For $i = 1, \ldots, m$, let E_i denote the event that the ith clause is not satisfied by the random assignment. Since each clause has k literals,

$$\Pr[E_i] = \frac{1}{2^k}.$$

The event E_i is mutually independent of all of the events related to clauses that do not share variables with clause i. By hypothesis, each of the variables in clause i can appear in no more than $T = 2^{k-2}/k$ clauses. Since there are k variables in a clause, the degree of the dependency graph is bounded by $d \leqslant kT \leqslant 2^{k-2}$.

Proof. (Continued)

It follows that

$$4dp \leqslant 4 \cdot 2^{k-2} \cdot 2^{-k} < 1$$

so we can apply the Lovasz local lemma to conclude that

$$\Pr\left[\bigcup_{i=1}^{m} E_i^c\right] > 0$$

hence there is a satisfying assignment for the formula.