Minimum Cuts in Graphs

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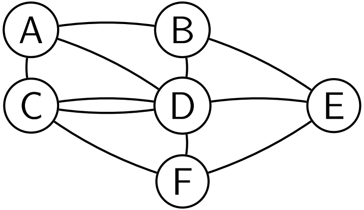
Let $G = (V, E)$ be a connected, undirected, loopfree multigraph with n vertices. A multigraph may contain multiple edges between two vertices, as the following example shows.



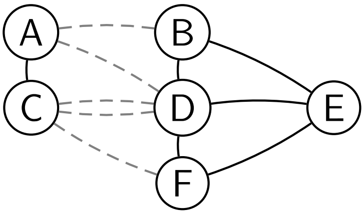
Definition

A **cut** in the multigraph $G = (V, E)$ is a partition of the vertex set V into two disjoint nonempty sets $V = V_1 \cup V_2$. An edge with one end in V_1 and the other in V_2 is said to **cross the cut**.

Remark

The term cut is chosen because the removal of the edges in a cut partitions the multigraph.

Example of a Cut



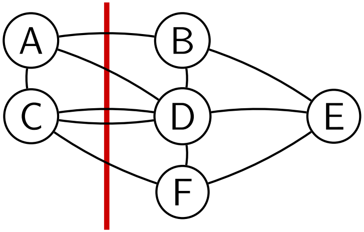
Example

If we partition $V = \{A, B, C, D, E, F\}$ into the sets

$$V_1 = \{A, C\} \text{ and } V_2 = \{B, D, E, F\},$$

then this cut has five crossing edges, and removing these edges yields a disconnected multigraph.

Size of a Cut



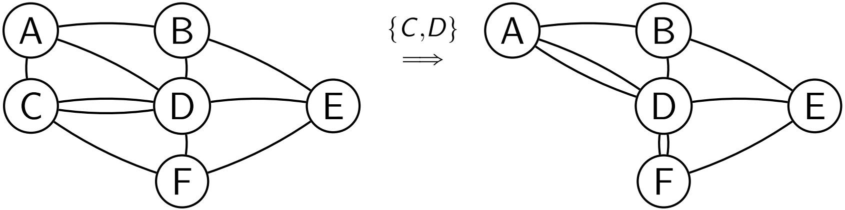
Definition

The **size** of the cut is given by the number of edges crossing the cut. The above example shows a cut of size 5.

Goal

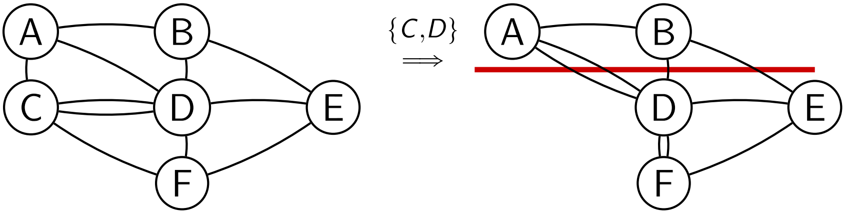
Determine the minimum size of a cut in a given multigraph G.

We describe a very simple randomized algorithm for this purpose. If e is an edge of a loopfree multigraph G, then the multigraph G/e is obtained from G by **contracting** the edge $e = \{x, y\}$, that is, we identify the vertices x and y and remove all resulting loops.



Remark

Note that any cut of G/e induces a cut of G.



Example

The cut $\{A, B\} \cup \{D, E, F\}$ in $G/\{C, D\}$ induces the cut $\{A, B\} \cup \{C, D, E, F\}$ in G. In general, the vertices that have been identified in G/e are in the same partition of G.

Observation

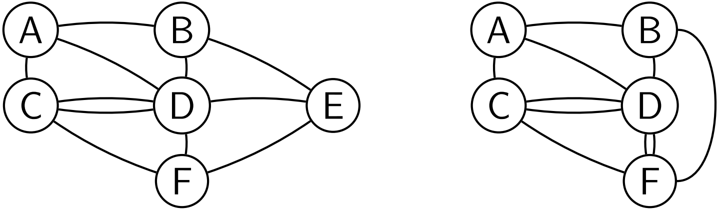
Remark

The size of the minimum cut of G/e is at least the size of the minimum cut of G, because all edges are kept.

Idea

- We can use successive contractions to estimate the size of the minimum cut of G.
- We can select uniformly at random one of the remaining edges and contract it until two vertices remain.
- The cut determined by this algorithm contains precisely the edges that have not been contracted.
- Counting the edges between the remaining two vertices yields an estimate of the size of the minimum cut of G.

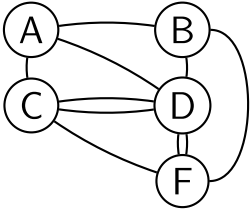
Example (1/4)

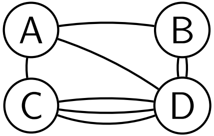


Step 1

Contract by $\{E, F\}$. Partition $\{\{A\}, \{B\}, \{C\}, \{D\}, \{E, F\}\}$.

Example (2/4)





Step 2

Contract by $\{D, F\}$. Partition $\{\{A\}, \{B\}, \{C\}, \{D, E, F\}\}$.

Step 3

Contract by $\{C, D\}$. Partition $\{\{A\}, \{B\}, \{C, D, E, F\}\}$.

Example (4/4)

Step 4

Contract by $\{B, D\}$. Partition $\{\{A\}, \{B, C, D, E, F\}\}$.

Contract(G)

Require: A connected loopfree multigraph $G = (V, E)$ with at least 2 vertices.

```
while |V| > 2 do
    Select e ∈ E uniformly at random;
    G := G/e;
end while
return |E|.
```

Ensure: An upper bound on the minimum cut of G.

Main Tool

Conditional Probability

$$\Pr[E \cap F] = \Pr[E|F]\Pr[F]$$

Exercise

Prove the following straightforward consequence of the previous formula

$$\Pr\left[\bigcap_{\ell=1}^{n} E_\ell\right] = \left(\prod_{m=2}^{n} \Pr\left[E_m \middle| \bigcap_{\ell=1}^{m-1} E_\ell\right]\right) \Pr[E_1].$$

If you expand the formula then you will immediately see the pattern.

Solution

Idea:

$$
\begin{aligned}
\Pr[E_n \cap E_{n-1} \cap \cdots \cap E_1] &= \Pr[E_n \mid E_{n-1} \cap \cdots \cap E_1]\Pr[E_{n-1} \cap \cdots \cap E_1] \\
\Pr[E_{n-1} \cap E_{n-2} \cap \cdots \cap E_1] &= \Pr[E_{n-1} \mid E_{n-2} \cap \cdots \cap E_1]\Pr[E_{n-2} \cap \cdots \cap E_1] \\
\Pr[E_{n-2} \cap E_{n-3} \cap \cdots \cap E_1] &= \Pr[E_{n-2} \mid E_{n-3} \cap \cdots \cap E_1]\,\Pr[E_{n-1} \cap \cdots \cap E_1] \\
&\vdots \\
\Pr[E_2 \cap E_1] &= \Pr[E_2 \mid E_1]\,\Pr[E_1]
\end{aligned}
$$

Rigorous proof: Induction.

Idea:

$$
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\Pr[E_n \cap E_{n-1} \cap \cdots \cap E_1] &= \Pr[E_n \mid E_{n-1} \cap \cdots \cap E_1] \Pr[E_{n-1} \cap \cdots \cap E_1] \\
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$$\vdots$$
$$\Pr[E_2 \cap E_1] = \Pr[E_2 \mid E_1]\Pr[E_1]$$

Rigorous proof: Induction.

Caution

Suppose that the multigraph has a uniquely determined minimum cut. If the algorithm selects in this case *any* edge crossing this cut, then the algorithm will fail to produce the correct result. The analysis is largely guided by this observation.

Exercise

Give an example of a connected, loopfree multigraph with at least four vertices that has a uniquely determined minimum cut.

Number of Iterations

Remark

Let $G = (V, E)$ be a loopfree connected multigraph with $n = |V|$ vertices. Note that each contraction reduces the number of vertices by one, so the algorithm terminates after $n - 2$ steps.

Analysis (1/4)

Suppose that C is a particular minimum cut of G. Let E_i denote the event that the algorithm selects in the ith step an edge that does not cross the cut C. Therefore, the probability that no edge crossing the cut C is ever picked during an execution of the algorithm is

$$\Pr\left[\bigcap_{j=1}^{n-2} E_j\right].$$

This probability can be calculated by

$$\Pr\left[\bigcap_{m=1}^{n-2} E_m\right] = \left(\prod_{m=2}^{n-2} \Pr\left[E_m \middle| \bigcap_{\ell=1}^{m-1} E_\ell\right]\right) \Pr[E_1].$$

Analysis (2/4)

Suppose that the size of the minimum cut is k.

This means that the degree of each vertex is at least k, hence there exist at least $kn/2$ edges.

The probability to select an edge crossing the cut C in the first step is at most $k/(kn/2) = 2/n$. Consequently,

$$\Pr[E_1] \geqslant 1 - \frac{2}{n} = \frac{n-2}{n}.$$

Similarly, at the beginning of the mth step, with $m \geqslant 2$, there are $n - m + 1$ remaining vertices.

The minimum cut is still at least k, hence the multigraph has at this stage at least $k(n - m + 1)/2$ edges. Assuming that none of the edges crossing C was selected in an earlier step, the probability to select an edge crossing the cut C is $2/(n - m + 1)$.
It follows that

$$\Pr\left[E_m \middle| \bigcap_{j=1}^{m-1} E_j\right] \geqslant 1 - \frac{2}{n - m + 1} = \frac{n - m - 1}{n - m + 1}.$$

Applying these lower bounds to the iterated conditional probabilities yields the result:

$$\Pr\left[\bigcap_{j=1}^{n-2} E_j\right] \geqslant \prod_{m=1}^{n-2} \left(\frac{n-m-1}{n-m+1}\right)$$

In other words, we have

$$\Pr\left[\bigcap_{j=1}^{n-2} E_j\right] \geqslant \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-4}{n-2}\right)\cdots\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{3}\right)$$

$$= \frac{2}{n(n-1)} = \binom{n}{2}^{-1}$$

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In conclusion, we have shown that the contraction algorithm yields the correct answer with probability at least $\Omega(1/n^2)$.

Number of Repetitions

Main Tool

Useful Inequality

$$1 + x \leqslant e^x.$$

Indeed, consider the function $f(x) = e^x - 1 - x$.

It has the derivative $f'(x) = e^x - 1$.

We have $f'(x) = 0$ if and only if $x = 0$.

The function $f(x)$ has a (global) minimum at $x = 0$, since $f''(0) = e^0 = 1 > 0$. We can conclude that $f(x) \geqslant f(0) = 0$.

Corollary

Consequently, for all positive integers n, we have

$$\left(1+\frac{x}{n}\right)^n \leqslant (e^{x/n})^n = e^x.$$

The probability that the algorithm fails to produce the correct result in one execution is

$$\Pr[\text{failure}] \leqslant (1 - 2/n^2).$$

Recall that for independent event E and F, the probability is given by $\Pr[E \cap F] = \Pr[E]\Pr[F]$. Therefore, if we execute the algorithm $n^2/2$ times, then the probability that the repeated executions will never reveal the correct size of the minimum cut is at most

$$\left(1 - \frac{2}{n^2}\right)^{n^2/2} \leqslant e^{-1}.$$

Analysis

If we repeat the algorithm $\frac{n^2 \ln n}{2}$ times, then the probability of obtaining an incorrect size of the minimum cut is at most

$$\left(1 - \frac{2}{n^2}\right)^{\frac{n^2 \ln n}{2}} \leqslant e^{-\ln n} = \frac{1}{n}.$$

We can conclude that repeating the contraction algorithm $O(n^2 \log n)$ times yields the correct size of the minimum cut with high probability.

FastCut

Motivation

- Assuming that the input multigraph has just a single minimum cut C, then Karger's minimum cut algorithm fails in a single run if and only if it contracts an edge of the minimum cut C.
- Selecting an edge crossing the cut C is more likely towards a later stage of the algorithm rather than at the beginning.

Idea

- We can subdivide the node contractions into different phases
- Early phases need fewer repetitions (restarts), since they are less likely to err
- Later phases need more repetitions (restarts), since this is where the errors are likely to happen
- We use recursion.

Contract(G, t)

Require: A connected loopfree multigraph $G = (V, E)$ with at least t vertices.

1: **while** $|V| > t$ **do**
2: **Select** $e \in E$ **uniformly at random**;
3: $\mathrm{G} := \mathrm{G}/\mathrm{e}$;
4: **end while**

Idea

Stop when t nodes are reached.

As in the case $t = 2$, we have

$$\Pr\Big[\bigcap_{j=1}^{n-t} E_j\Big] \geqslant \prod_{j=1}^{n-t}\left(\frac{n-m-1}{n-m+1}\right)$$

We get the lower bound

$$\begin{aligned}\Pr\Big[\bigcap_{j=1}^{n-t} E_j\Big] &\geqslant \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\cdots\left(\frac{t}{t+2}\right)\left(\frac{t-1}{t+1}\right)\\ &= \frac{\binom{t}{2}}{\binom{n}{2}}\end{aligned}$$

Proposition

If $t \geqslant \frac{n}{\sqrt{2}} + 1$, then $\binom{t}{2} / \binom{n}{2} \geqslant 1/2$.

The function

$$\binom{t}{2} \Big/ \binom{n}{2} = \frac{t(t-1)}{n(n-1)}$$

is increasing in t. Substituting $t = \frac{n}{\sqrt{2}} + 1$ yields

$$\frac{t(t-1)}{n(n-1)} = \frac{\left(\frac{n}{\sqrt{2}} + 1\right)\frac{n}{\sqrt{2}}}{n(n-1)} \geqslant \frac{\frac{n^2}{2} - \frac{n}{2}}{n(n-1)} = \frac{1}{2}$$

FastCut(G)

Require: A connected loopfree multigraph $G = (V, E)$.

$n = |V|$.
return mincut(G) **if** $n \leqslant 6$ // brute force computation
$t = \lceil n/\sqrt{2} + 1 \rceil$.
$G_1 = Contract(G, t)$.
$G_2 = Contract(G, t)$.
return min(FastCut(G_1), FastCut(G_2));

Proposition

FastCut runs in $O(n^2 \log n)$ time.

Proof.

The algorithm Contract uses $O(n^2)$ time to reduce a multigraph with n vertices down to 2 vertices. Thus, reducing it twice to t vertices can certainly be done in $O(n^2)$ time. The time $T(n)$ of FastCut satisfies the recurrence

$$T(n) = 2T\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right) + O(n^2).$$

The solution to this recurrence satisfies $T(n) = O(n^2 \log n)$. $\square$

Success Probability

Proposition

FastCut finds a minimum cut with probability $\Omega(1/\log n)$.

Proof.

We already showed that minimum cut C survives the contractions from n to $n/\sqrt{2}+1$ vertices with probability $1/2$ or more.

Let $P(n)$ denote the probability that FastCut succeeds in finding a minimum cut in a multigraph with n vertices. Then

$$P(n) \geqslant 1 - \left(1 - \frac{1}{2} P\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right)\right)^2 .$$

Proof (Continued).

We will solve this recurrence by making a change of variables. The depth of the recursion is $k = O(\log n)$. Let $p(k)$ denote a lower bound on the success probability at level k. Then

$$p(0) = 1$$

and (from the previous inequality)

$$p(k+1) = p(k) - \frac{p(k)^2}{4}.$$

Proof (Continued).

We can solve this by setting $q(k) = 4/p(k) - 1$, which amounts to

$$p(k) = 4/(q(k) + 1).$$

Substituting into the previous equation yields

$$q(k+1) = q(k) + 1 + \frac{1}{q(k)}.$$

By induction, we have

$$k < q(k) < k + H_{k-1} + 4,$$

where $H_{k-1} = 1 + 1/2 + \cdots + 1/(k-1)$. It follows that

$$q(k) = k + \Theta(\log k).$$

Proof (Continued).

Since $q(k) = k + \Theta(\log k)$ and by definition

$$p(k) = \frac{4}{q(k)+1} = \frac{4}{k + \Theta(\log k) + 1}.$$

We have

$$\lim_{k\to\infty} p(k)/(1/k) = \lim_{k\to\infty} \frac{4k}{k + \Theta(\log k) + 1} = 4.$$

It follows that $p(k) = \Theta(1/k)$. Since $p(k)$ was the lower bound to $P(n)$ with recursion depth $k = \Theta(\log n)$, we can conclude that

$$P(n) \geqslant p(\log n) = \Omega(1/\log n). \quad \square$$