Andreas Klappenecker

Texas A&M University

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Definition

Let X be a discrete random variable defined on a probability space with probability measure Pr. Assume that X has non-negative integer values. The **probability generating function** of X is defined by

$$G_X(z) = \mathsf{E}[z^X] = \sum_{k=0}^{\infty} \mathsf{Pr}[X=k]z^k.$$

This series converges for all z with $|z| \leq 1$.

Expectation

The expectation value can be expressed by

$$\mathsf{E}[X] = \sum_{k=1}^{\infty} k \, \mathsf{Pr}[X = k] = G'_X(1), \tag{1}$$

where $G'_X(z)$ denotes the derivative of $G_X(z)$.

Indeed,
$$G'_X(z) = \sum_{k=0}^{\infty} k \Pr[X = k] z^{k-1} = \sum_{k=1}^{\infty} k \Pr[X = k] z^{k-1}$$
.

Second Moment

Second Moment

$$\mathsf{E}[X^2] = G''_X(1) + G'_X(1)$$

Indeed,

$$G'_X(z) = \sum_{k=1}^{\infty} k \operatorname{Pr}[X = k] z^{k-1}$$

and

$$G_X''(z) = \sum_{k=2}^{\infty} k(k-1) \Pr[X = k] z^{k-2} = \sum_{k=2}^{\infty} (k^2 - k) \Pr[X = k] z^{k-1}$$

Variance

$$Var[X] = E[X^2] - E[X]^2$$

= $G''_X(1) + G'_X(1) - G'_X(1)^2$.

Example

Let X be a random variable that has Bernoulli distribution with parameter p. The probability generating function is given by

$$G_X(z) = (1-p) + pz.$$

Hence $G'_X(z) = p$, and G''(z) = 0. We obtain $E[X] = G'_X(1) = p$ and

$$\mathsf{Var}[X] = G_X''(1) + G_X'(1) - G_X'(1)^2 = 0 + p - p^2 = p(1-p).$$

Geometric Random Variables Example

The probability generating function of a geometrically distributed random variable X is

$$G(z) = \sum_{k=1}^{\infty} p(1-p)^{k-1} z^k = pz \sum_{k=0}^{\infty} (1-p)^k z^k = \frac{pz}{1-(1-p)z}.$$

Some calculus shows that

$$G'(z) = rac{p}{(1-(1-p)z)^2}, \qquad G''(z) = rac{2p(1-p)}{(1-(1-p)z)^3}.$$

Therefore, the expectation value is $E[X] = G'_X(1) = 1/p$. The variance is given by

$$\operatorname{Var}[X] = G''(1) + G'(1) - G'(1)^2 = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

Proposition

Let X_1, \ldots, X_n be independent $\mathbb{Z} \ge$ -valued random variables with probability generating functions $G_{X_1}(z), \ldots, G_{X_n}(z)$. The probability generating function of $X = X_1 + \cdots + X_n$ is given by the product

$$G_X(z) = \prod_{k=1}^n G_{X_k}(z).$$

Proof.

It suffices to show this for two random variables X and Y. The general case can be established by a straightforward induction proof.

$$G_X(z)G_Y(z) = \left(\sum_{k=0}^{\infty} \Pr[X=k]z^k\right) \left(\sum_{k=0}^{\infty} \Pr[Y=k]z^k\right)$$
$$= \sum_{k=0}^{\infty} z^k \left(\sum_{\ell=0}^k \Pr[X=\ell]\Pr[Y=k-\ell]\right)$$
$$= \sum_{k=0}^{\infty} z^k \left(\sum_{\ell=0}^k \Pr[X=\ell,Y=k-\ell]\right)$$
$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^k \Pr[X+Y=k]z^k = G_{X+Y}(z)$$

Example

Recall that the Bernoulli distribution with parameter p has generating function (1 - p) + pz. If X_1, \ldots, X_n are independent random variables that are Bernoulli distributed with parameter p, then $X = X_1 + \cdots + X_n$ is, by definition, binomially distributed with parameters n and p. The generating function of X is

$$G_X(z) = ((1-p) + pz)^n = \sum_{k=0}^n \binom{n}{k} (1-p)^{n-k} p^k z^k.$$

Example (Continued.)

We have

$$G_X'(z)=np((1-p)+pz)^{n-1}$$

The expected value is given by

$$\mathsf{E}[X] = G'_X(1) = np.$$

Binomial Distribution

Example (Continued.)

We have

$$G'_X(z) = np((1-p) + pz)^{n-1}$$

 and

$$G''_X(z) = n(n-1)p^2((1-p)+pz)^{n-2}$$

The expected value is given by

$$Var[X] = G''_X(1) + G'_X(1) - G'_X(1)^2$$

= $(n^2 - n)p^2 + np - n^2p^2$
= $-np^2 + np = np(1 - p)$

Proposition

Let X and Y be discrete random variables with probability generating functions $G_X(z)$ and $G_Y(z)$, respectively. Then the probability generating function

$$G_X(z) = G_Y(z)$$

if and only if the probability distributions

$$\Pr[X = k] = \Pr[Y = k]$$

for all integers $k \ge 0$.

If the probability distributions are the same, then evidently $G_X(z) = G_Y(z)$.

Conversely, suppose that the generating functions $G_X(z)$ and $G_Y(z)$ are the same. Since the radius of convergence is at least 1, we can expand the two generating functions into power series

$$G_X(z) = \sum_{k=0}^{\infty} \Pr[X = k] z^k$$
$$G_Y(z) = \sum_{k=0}^{\infty} \Pr[Y = k] z^k$$

These two power series must have identical coefficients, since the generating functions are the same. Therefore, $\Pr[X = k] = \Pr[Y = k]$ for all $k \ge 0$, as claimed.

Number of Inversions

Definition

Let $(a_1, a_2, ..., a_n)$ be a permutation of the set $\{1, 2, ..., n\}$. The pair (a_i, a_j) is called an **inversion** if and only if i < j and $a_i > a_j$.

Example

The permutation (3, 4, 1, 2) has the inversions

 $\{(3,1),(3,2),(4,1),(4,2)\}.$

Number of Inversions

Definition

Let $I_n(k)$ denote the **number of permutations** on *n* points with *k* inversions.

Example

We have $I_n(0) = 1$, since only the identity has no inversions.

Example

We have $I_n(1) = n - 1$. Indeed, a permutation π can have a single inversion if and only if π is equal to a transposition of neighboring elements (k + 1, k) for some k in the range $1 \le k \le n - 1$.

Number of Inversions

Example

Since no permutation can have more than $\binom{n}{2}$ inversions, we have

$$I_n(k) = 0$$
 for all $k > \binom{n}{2}$.

Example

By reversal of the permutations, we have the symmetry

$$I_n\left(\binom{n}{2}-k\right)=I_n(k)$$

Suppose that we choose permutations π uniformly at random from the symmetric group S_n .

Let X_n denote the random variable on S_n that assigns a permutation π its number of inversions. Then the probability generating function

$$G_{X_n}(z) = \sum_{k=0}^{\binom{n}{2}} \Pr[X_n = k] z^k$$

is given by

$$G_{X_n}(z) = \sum_{k}^{\binom{n}{2}} \frac{I_n(k)}{n!} z^k.$$

Question

Can we relate the generating functions $G_{X_n}(z)$ and $G_{X_{n-1}}(z)$?

Question

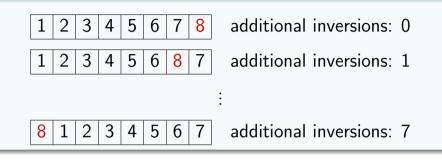
Can we relate the generating functions $G_{X_n}(z)$ and $G_{X_{n-1}}(z)$?

Observation

Suppose that we have a permutation π_{n-1} on $\{1, 2, ..., n-1\}$. If we insert the element *n* at position *j* with $1 \le j \le n$, then we get an additional n-j inversions.



Example



Observation

We need to insert n uniformly at random to obtain a uniformly distributed permutation on n elements from uniformly distributed permutations on n-1 elements.

Proposition

$$G_{X_n}(z) = \begin{cases} rac{(1+z+z^2+\cdots+z^{n-1})}{n}G_{X_{n-1}}(z) & \textit{if } n>1, \ 1 & \textit{if } n=1. \end{cases}$$

Corollary

$$G_{X_n}(z) = \frac{1}{n!} \prod_{k=1}^n (1 + z + z^2 + \dots + z^{n-1})$$
$$= \prod_{k=1}^n \frac{1 - z^k}{k(1 - z)} = \frac{1}{n!} \prod_{k=1}^n \frac{1 - z^k}{1 - z}$$

Factorization: Expected Value

In other words, the generating function $G_{X_n}(z)$ is the product of generating functions of discrete uniform random variables U_k on $\{0, 1, \ldots, k-1\}$,

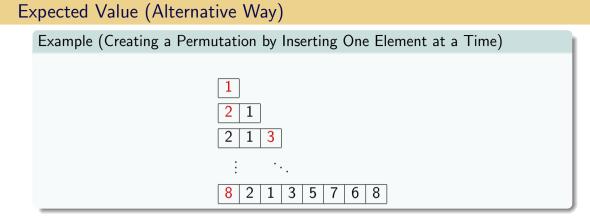
$$G_{X_n}(z)=\prod_{k=1}^n G_{U_k}(z), \quad ext{where} \quad G_{U_k}(z)=rac{1+z+\cdots+z^{k-1}}{k}.$$

By the product rule for n functions, we have

$$G'_{U_n}(z) = \prod_{k=1}^n G_{U_k}(z) \sum_{\ell=1}^n \frac{\frac{1}{n}(1+2z+3z^2+\cdots+(\ell-1)z^{\ell-2})}{G_{U_\ell}(z)}.$$

Then

$$\mathsf{E}[X_n] = G'_{U_n}(1) = \sum_{k=1}^n \frac{k-1}{2} = \frac{n(n-1)}{4}.$$



Observation

$$X_n = U_1 + U_2 + \dots + U_n$$
$$\mathsf{E}[X_n] = \mathsf{E}[U_1] + \mathsf{E}[U_2] + \dots + \mathsf{E}[U_n]$$

Proposition

$$\mathsf{E}[X_n] = \sum_{k=1}^n \mathsf{E}[U_k] = \sum_{k=1}^n \frac{k-1}{2} = \frac{n(n-1)}{4}.$$

Variance

Proposition

$$Var[X_n] = rac{2n^3 + 3n^2 - 5n}{72}.$$

Proof.

Since $X_n = U_1 + U_2 + \cdots + U_n$ and the U_k are mutually independent, we get

$$\operatorname{Var}[X_n] = \sum_{k=1}^n \operatorname{Var}[U_n] = \sum_{k=1}^n \frac{k^2 - 1}{12} = \frac{1}{12} \left(\sum_{k=1}^n k^2 - n \right)$$
$$= \frac{1}{12} \left(\frac{2n^3 + 3n^2 + n}{6} - n \right)$$
$$= \frac{2n^3 + 3n^2 - 5n}{72}.$$