Graphs with Large Girth and Large Chromatic Number

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We denote by $\chi(G)$ the chromatic number of a graph G, by $\alpha(G)$ the independence number of G, and by $\mathrm{girth}(G)$ the girth of G.

Theorem 1 (Erdős)**.** *For all nonnegative integers k and ℓ there exists a graph G with $\mathrm{girth}(G) > \ell$ and $\chi(G) > k$.*

Proof. Let $G = G(n,p)$ be a random graph with n vertices, where for each pair of vertices an edge is chosen with probability p independently of other edges. Let us chose θ such that $\theta < 1/\ell$ and $p = n^{\theta-1}$.

We can specify a cycle with i edges by selecting a sequence of i vertices, namely the sequence $(v_1, v_2, \cdots, v_i)$ of i vertices specifies the cycle consisting of the i edges (v_1, v_2), (v_2, v_3), $\ldots (v_{i-1}, v_i), (v_i, v_1)$.

There are $n(n-1)\cdots(n-i+1)$ sequences of i vertices in G, but not all of them specify distinct cycles. The i cyclically rotated sequences

$$(v_{1+a}, v_{2+a}, \cdots, v_i, v_1, \ldots, v_a)$$

with $a \in \{0, \cdots, i-1\}$, and the i mirrored sequences

$$(v_a, v_{a-1}, \ldots, v_1, v_i, \ldots, v_{1+a})$$

all specify the same cycle of edges. Therefore, there are $n(n-1)\cdots(n-i+1)/2i$ sequences of vertices that specify different cycles.

Let X denote the number of cycles in G of size at most ℓ. Then

$$\mathrm{E}[X] = \sum_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i.$$

Since $(n-i+1)p < \cdots < np < n^\theta$, we get

$$\mathrm{E}[X] \leq \sum_{i=3}^{\ell} n^{\theta i} = o(n).$$

It follows from Markov's inequality that

$$\Pr[X \geq n/2] = o(1).$$

Let us choose $x = \lceil \frac{3}{p} \ln n \rceil$. By a union bound,

$$\Pr[\alpha(G) \geq x] \leq \binom{n}{x} (1-p)^{\binom{x}{2}}.$$

Notice that $\binom{n}{x} \leq n^x$. Furthermore, using the inequality $1 - y \leq e^{-y}$, we get $(1-p)^{\binom{x}{2}} \leq e^{-p\binom{x}{2}}$. Therefore,

$$\Pr[\alpha(G) \geq x] \leq n^x e^{-p\binom{x}{2}} = \left(ne^{-p(x-1)/2}\right)^x = o(1).$$

Let n be sufficiently large such that $\Pr[X \geq n/2] < 1/2$ and $\Pr[\alpha(G) \geq x] < 1/2$. Then there exists a G with less than $n/2$ cycles of length at most ℓ and with $\alpha(G) < 3n^{1-\theta} \ln n$. Remove from G a vertex from each cycle of length at most ℓ. This give a graph G^* with at least $n/2$ vertices.

Since an independent set in G^* is an independent set in G, we have $\alpha(G^*) \leq \alpha(G)$.

If $h = \alpha(G^*)$, then no class of colors in G^* can have more than h vertices. Therefore, $\chi(G^*) \geq |G^*|/\alpha(G^*)$. It follows that

$$\chi(G) \geq \chi(G^*) \geq \frac{|G^*|}{\alpha(G^*)} \geq \frac{n/2}{3n^{1-\theta} \ln n} = \frac{n^\theta}{6 \ln n}.$$

Now simply choose n sufficiently large such that $\frac{n^\theta}{6 \ln n} \geq k$. $\square$

This proof is slightly expanded version of the proof given on pages 38–39 in [N. Alon and J. Spencer, *The Probabilistic Method*, 2nd edition, Wiley-Interscience, 2000].