Tensor Products

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Tensor Product: A Wish List

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$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \quad (1)$$

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and the **balancing relations**

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for each v, v_1, v_2 in V, each w, w_1, w_2 in W, and each complex number c.

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for each v, v_1, v_2 in V, each w, w_1, w_2 in W, and each complex number c.

Of course, we need to establish the existence of such a product $\otimes$.

Warning

We emphasize that not every element of $V \otimes W$ is of the form $v \otimes w$ for some $v \in V$ and $w \in W$!

However, every element of $V \otimes W$ can be expressed as a sum $\sum_{i,j} v_i \otimes w_j$ of such tensor products, with $v_i \in V$ and $w_j \in W$.

Construction of the Tensor Product

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$$(v, w_1 + w_2) - (v, w_1) - (v, w_2),$$
$$c(v, w) - (cv, w), \quad c(v, w) - (v, cw),$$

for $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $c \in \mathbf{C}$.

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for $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $c \in \mathbf{C}$. We define the **tensor product** $V \otimes W$ to be the quotient space A/B. The image of the element (v, w) of A in $V \otimes W$ is denoted by $v \otimes w$.

Recall that the vector space A that is spanned by linear combinations of the elements

$$(v, w), \qquad v, w \in \mathbf{C}^2.$$

Let u_1 and u_2 be vectors in A. We consider them the same if and only if they differ by a vector in B. We define

$$u_1 \equiv u_2 \pmod{B}$$

if and only if

$$u_1 - u_2 \in B.$$

Then $\equiv$ is an equivalence relation; A/B is the set of equivalence classes.

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Since $(v_1 + v_2, w) - (v_1, w) - (v_2, w) \in B$, we have

$$(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w = 0$$

in $A/B = V \otimes W$. Other rules: similar!

Observation

Let B_V be a basis of V and B_W be a basis of W. Then

$$\{x \otimes y \mid x \in B_V, y \in B_W\}$$

is a basis of $V \otimes W$.

In particular, $\dim V \otimes W = (\dim V)(\dim W)$.

Example

Let $\mathbf{C}^2$ the vector space with basis $|0\rangle$ and $|1\rangle$.

Then $\mathbf{C}^2 \otimes \mathbf{C}^2$ is a 4-dimensional vector space with basis

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We will identify $\mathbf{C}^2 \otimes \mathbf{C}^2$ with $\mathbf{C}^4$ by the following isomorphism

$$|x\rangle \otimes |y\rangle \mapsto |xy\rangle$$

for $x, y \in \{0, 1\}$.

Example

The vector space $\mathbf{C}^2 \otimes \mathbf{C}^2$ contains the vector

$$\frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle.$$

We cannot write it in the form

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle),$$

since this would mean that

$$ac \neq 0, \quad ad = 0, \quad bc = 0, \quad bd \neq 0.$$

Tensor Product of Linear Maps

Let V and W be finite-dimensional vector spaces. Let $\{e_1, \ldots, e_m\}$ be a basis of V and $\{f_1, \ldots, f_n\}$ be a basis of W.

Suppose that A is a linear map on V, and B is a linear map on W. Let $A \otimes B$ denote the linear map on $V \otimes W$, which is determined by

$$(A \otimes B)(e_i \otimes f_j) = Ae_i \otimes Bf_j.$$

This uniquely determines the values of $A \otimes B$ on other elements of $V \otimes W$ because the elements $e_i \otimes f_j$ are a basis.

Suppose that the linear map A and B are given by the matrices

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}.$$

Then $A \otimes B$ is given by the matrix

$$\begin{pmatrix} a_{00}\,b_{00} & a_{00}\,b_{01} & a_{01}\,b_{00} & a_{01}\,b_{01} \\ a_{00}\,b_{10} & a_{00}\,b_{11} & a_{01}\,b_{10} & a_{01}\,b_{11} \\ a_{10}\,b_{00} & a_{10}\,b_{01} & a_{11}\,b_{00} & a_{11}\,b_{01} \\ a_{10}\,b_{10} & a_{10}\,b_{11} & a_{11}\,b_{10} & a_{11}\,b_{11} \end{pmatrix}.$$