The Deutsch-Josza Algorithm

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Given

A black-box Boolean function $f\colon \mathbf{F}_2^n \to \mathbf{F}_2$.

The function is either **constant** or **balanced** (meaning that half of the inputs evaluate to 0 and the other half to 1).

Goal

Determine whether f is constant or balanced using as few calls to the black-box function as possible.

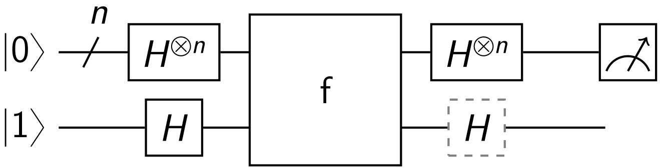
Classical Solution

Any deterministic classical solution to the problem requires at least

$$2^{n-1} + 1 \quad \text{queries}$$

to the black-box function f, since 2^{n-1} or fewer oracle calls would not allow us to discriminate between constant and balanced functions with **certainty**.

Quantum Solution



First step (after Hadamards).

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \mathbf{F}_2^n} |x\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right)$$

Second step (apply f).

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \mathbf{F}_2^n} (-1)^{f(x)}|x\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right)$$

Third step (after Hadamards).

$$\frac{1}{2^n}\sum_{x\in\mathbf{F}_2^n}(-1)^{f(x)}\sum_{z\in\mathbf{F}_2^n}(-1)^{x\cdot z}|z\rangle\otimes\left(\frac{1}{\sqrt{2}}|0\rangle-\frac{1}{\sqrt{2}}|1\rangle\right)$$

$$=\frac{1}{2^n}\sum_{z\in\mathbf{F}_2^n}\left(\sum_{x\in\mathbf{F}_2^n}(-1)^{f(x)+x\cdot z}\right)|z\rangle\otimes\left(\frac{1}{\sqrt{2}}|0\rangle-\frac{1}{\sqrt{2}}|1\rangle\right)$$

Final Step: Measurement

The probability to observe $z = 00\cdots 0$ is

$$\left(\frac{1}{2^n} \sum_{x \in \mathbf{F}_2^n} (-1)^{f(x) + x \cdot 0} \right)^2 = \begin{cases} 1 & \text{if } f \text{ is constant} \\ 0 & \text{if } f \text{ is balanced} \end{cases}$$

Conclusion

One can solve the Deutsch-Josza problem with a single query on a quantum computer, whereas $2^{n-1} + 1$ queries are needed on a classical computer.