Continued Fractions

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Let m be a nonnegative integer. A **finite continued fraction** is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_m}}}}}$$

It is notationally more convenient to denote this expression in the form

$$[a_0, a_1, \ldots, a_m].$$

The terms $a_0, \ldots, a_m$ are called **partial quotients** of the continued fraction.

Example 1. We note that

$$[a_0] = \frac{a_0}{1}, \qquad [a_0, a_1] = \frac{a_1 a_0 + 1}{a_1} = a_0 + \frac{1}{a_1},$$

and

$$[a_0, a_1, \ldots, a_{m-1}, a_m] = [a_0, a_1, \ldots, a_{m-2}, a_{m-1} + \frac{1}{a_m}].$$

We may assume that the partial quotients $a_1, \ldots, a_m$ of the continued fraction $[a_0, a_1, \ldots, a_m]$ are positive.

We closely follow in these notes the excellent exposition of continued fractions given in [S. Lang "Introduction to Diophantine Approximations", Springer Verlag, 2nd edition, 1995].

Convergents. If $\alpha = [a_0, \ldots, a_m]$ is a continued fraction, then we call

$$[a_0, \ldots, a_k]$$

the k^{th} **principal convergent** to α (or the k^{th}**convergent** to α for short), where k is an integer in the range $0 \leq k \leq m$.

Theorem 2. *Let $\alpha = [a_0, \ldots, a_m]$ be a continued fraction such that the partial quotients $a_1, \ldots, a_m$ are positive. For all k in the range $0 \leq k \leq m$, we define numbers p_k and q_k by*

$$\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

Then the k^{th} convergent of α is given by

$$\frac{p_k}{q_k} = [a_0, \ldots, a_k].$$

Proof. For $k = 0$ this follows directly from the definitions.

Suppose that the theorem holds for $k < m$. Our goal is to show that the $(k+1)^{\text{th}}$ convergent is of the form p_{k+1}/q_{k+1}.

Equation (1) shows that the numbers p_{k+1} and q_{k+1} can be expressed in terms of the numbers p_k, p_{k-1} and q_k, q_{k-1}, respectively. More explicitly,

$$\begin{pmatrix} p_{k+1} & p_k \\ q_{k+1} & q_k \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} \begin{pmatrix} a_{k+1} & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Recall that

$$[a_0, a_1, \ldots, a_{k-1}, a_k, a_{k+1}] = [a_0, a_1, \ldots, a_{k-1}, a_k + \frac{1}{a_{k+1}}].$$

By induction hypothesis, the right hand side can be expressed in the form

$$\begin{aligned} [a_0, a_1, \ldots, a_{k-1}, a_k + \tfrac{1}{a_{k+1}}] &= \frac{\left(a_k + \frac{1}{a_{k+1}}\right) p_{k-1} + p_{k-2}}{\left(a_k + \frac{1}{a_{k+1}}\right) q_{k-1} + q_{k-2}} \\ &= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\ &= \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}} = \frac{p_{k+1}}{q_{k+1}}, \end{aligned}$$

where we have used the recurrence (2). Therefore, the theorem follows by induction. □

Corollary 3. *The convergents satisfy for all positive integers k the equation*

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}.$$

Proof. Taking determinants in equation (1) yields the claim. □

Corollary 4. *For $k \geq 1$, we have*

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}.$$

If a_0 is an integer and $a_1, \ldots, a_m$ are positive integers, then $[a_0, \ldots, a_m]$ is called a **simple continued fraction**.

Theorem 5. *The convergents of simple continued fractions have the following properties:*

(i) If $k > 1$, then $q_k \geq q_{k-1} + 1$; in particular, $q_k \geq k$.
(ii) $\frac{p_{2k+1}}{q_{2k+1}} < \frac{p_{2k-1}}{q_{2k-1}}$ and $\frac{p_{2k}}{q_{2k}} > \frac{p_{2k-2}}{q_{2k-2}}$
(iii) Every convergent of a simple continued fraction is a reduced fraction.

Proof. See S. Lang "Introduction to Diophantine Approximations", Springer Verlag, Chapter 1. □

Continued Fraction Algorithm. Let α_0 be a positive rational number. Our goal is to find a simple continued fraction representing α_0.

Set $a_0 = \lfloor \alpha_0 \rfloor$. For $k \geq 1$, we inductively define rational numbers α_k and their integral parts $a_k = \lfloor \alpha_k \rfloor$ by

$$\alpha_{k-1} = a_{k-1} + \frac{1}{\alpha_k} \tag{3}$$

assuming that $\alpha_{k-1} \neq a_{k-1}$.

This process stops after a finite number of steps. Indeed, suppose that $\alpha_{k-1} = a/b$, where a and b are coprime integers such that $b > 0$. Then

$$\frac{1}{\alpha_k} = \alpha_{k-1} - a_{k-1} = \frac{a - b\lfloor a/b \rfloor}{b}.$$

Since $c = a - b\lfloor a/b \rfloor$ is the remainder of the divison of a by b, we have $c < b$. Therefore, $\alpha_k = b/c$ is a rational number whose denominator is strictly less than the denominator of α_{k-1}.

Assuming that the process terminates after m iterations, it follows from equation (3) that $[a_0, \ldots, a_m]$ is a simple continued fraction representation of the input α_0.

Best Approximation. For a real number β, we denote by $\|\beta\|$ the distance between β and the nearest integer; put differently,

$$\|\beta\| = \min\{|\beta - n| \mid n \in \mathbf{Z}\}.$$

A **best approximation** to a real number α is a fraction p/q such that

$$\|q\alpha\| = |q\alpha - p|$$

and $\|q'\alpha\| > \|q\alpha\|$ holds for all q' in the range $1 \leq q' < q$.

Theorem 6. *The best approximations to α are the principal convergents to α. Moreover, if $n \geq 1$, then q_n is the smallest integer $q > q_{n-1}$ leading to an improved approximation $\|q\alpha\| < \|q_{n-1}\alpha\|$.*

Proof. Our first goal is to show that a best approximation is a convergent. Let a/b denote a reduced fraction with $b > 0$ such that a/b is a best approximation to α. In other words, we need to show that $a/b = p_n/q_n$ for some integer $n \geq 0$.

Suppose that $a/b < p_0/q_0 = a_0$. Then

$$|\alpha - a_0| < |\alpha - \frac{a}{b}| \leq |b\alpha - a|,$$

contradicting our assumption that a/b is a best approximation to α.

Suppose that $a/b > p_1/q_1$. Then

$$\left|\frac{a}{b} - \alpha\right| > \left|\frac{a}{b} - \frac{p_1}{q_1}\right| \geq \frac{1}{bq_1}.$$

Therefore,

$$|b\alpha - a| > \frac{1}{q_1} = \frac{1}{a_1} \geq |\alpha - a_0|,$$

contradicting our assumption that a/b is a best approximation to α.

Finally, suppose that a/b lies between p_{n-1}/q_{n-1} and p_{n+1}/q_{n+1}, but is not equal to either of these fractions. Then

$$\frac{1}{bq_{n-1}} \leq \left|\frac{a}{b} - \frac{p_{n-1}}{q_{n-1}}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{q_n q_{n-1}}.$$

Hence, $q_n < b$. On the other hand,

$$\frac{1}{nq_{n-1}} \leq \left|\frac{p_{n+1}}{q_{n+1}} - \frac{a}{b}\right| \leq \left|\alpha - \frac{a}{b}\right|,$$

whence

$$|q_n\alpha - p_n| < \frac{1}{q_{n+1}} \leq |b\alpha - a|,$$

which contradicts once again our assumption that a/b is a best approximation to α.

For the converse, see [S. Lang "Introduction to Diophantine Approximations", Springer Verlag, Chapter 1, Theorem 6, 2nd edition, 1995].

□

Corollary 7. *If a/b is a reduced fraction with $b > 0$ such that*

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{2b^2},$$

then a/b is a principal convergent to α.

Proof. It suffices to show that a/b is a best approximation to α. Let c/d be any fraction with $d > 0$ that is different from a/b such that

$$|d\alpha - c| \leq |b\alpha - a| < \frac{1}{2b}.$$

Then

$$\frac{1}{bd} \leq \left|\frac{c}{d} - \frac{a}{b}\right| \leq \left|\alpha - \frac{c}{d}\right| + \left|\alpha - \frac{a}{b}\right| < \frac{1}{2bd} + \frac{1}{2b^2} = \frac{b+d}{2b^2d}.$$

This implies that $b > d$. Therefore, a/b is a best approximation to α; hence, a/b is a principal convergant to α by the previous theorem. □