Tensor Products

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Motivation

Let V and W be vector spaces. The tensor product $V \otimes W$ is a vector space that is spanned by the elements $v \otimes w$ with v in V and w in W. It satisfies the following properties:

Left-Additivity: $(v_1+v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

Right-Additivity: $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

Balancing Relations: $c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$

for all complex scalars c, vectors v, v_1, v_2 in V, and w, w_1, w_2 in W.

Tensor Product Construction

Let V and W be vector spaces. Form a (giant!) vector space A of linear combinations of elements (v,w) with v in V and w in W.

Let B be the subspace of A which consists of linear combination of elements:

- $(v_1+v_2, w) - (v_1, w) - (v_2, w)$
- $(v, w_1+w_2) - (v, w_1) - (v, w_2)$
- $c(v,w) - (cv,w)$
- $c(v,w) - (v,cw)$

Form the quotient space A/B and denote the image of (v,w) in A/B by $v\otimes w$. Set $V\otimes W := A/B$.

Properties

Let B_V be a basis of V and B_W be a basis of W.

Then $\{ x \otimes y \mid x \text{ in } B_V \text{ and } y \text{ in } B_W \}$ is a basis of $V \otimes W$.

In particular, $\dim V \otimes W = (\dim V)(\dim W)$.

Universal Property

Let V and W be vector spaces. The tensor product $V \otimes W$ has the following universal property:

If B: V x W -> U is a bilinear map from (V,W) to a vector space U, then there exists a unique linear map b: $V \otimes W$-> U such that

$$B(v,w) = b(v,w)$$

holds for all v in V and w in W.

Caution!

Not all elements of $V \otimes W$ can be expressed in the form $v \otimes w$.

Problem: Consider $C^2 \otimes C^2$. Show that the vector $|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle$ cannot be expressed in the form $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$.

A vector in $V \otimes W$ of the form $v \otimes w$ is called separable. In quantum mechanics, non-separable state vectors are called entangled.