Teleporting Several Quantum Bits

Andreas Klappenecker

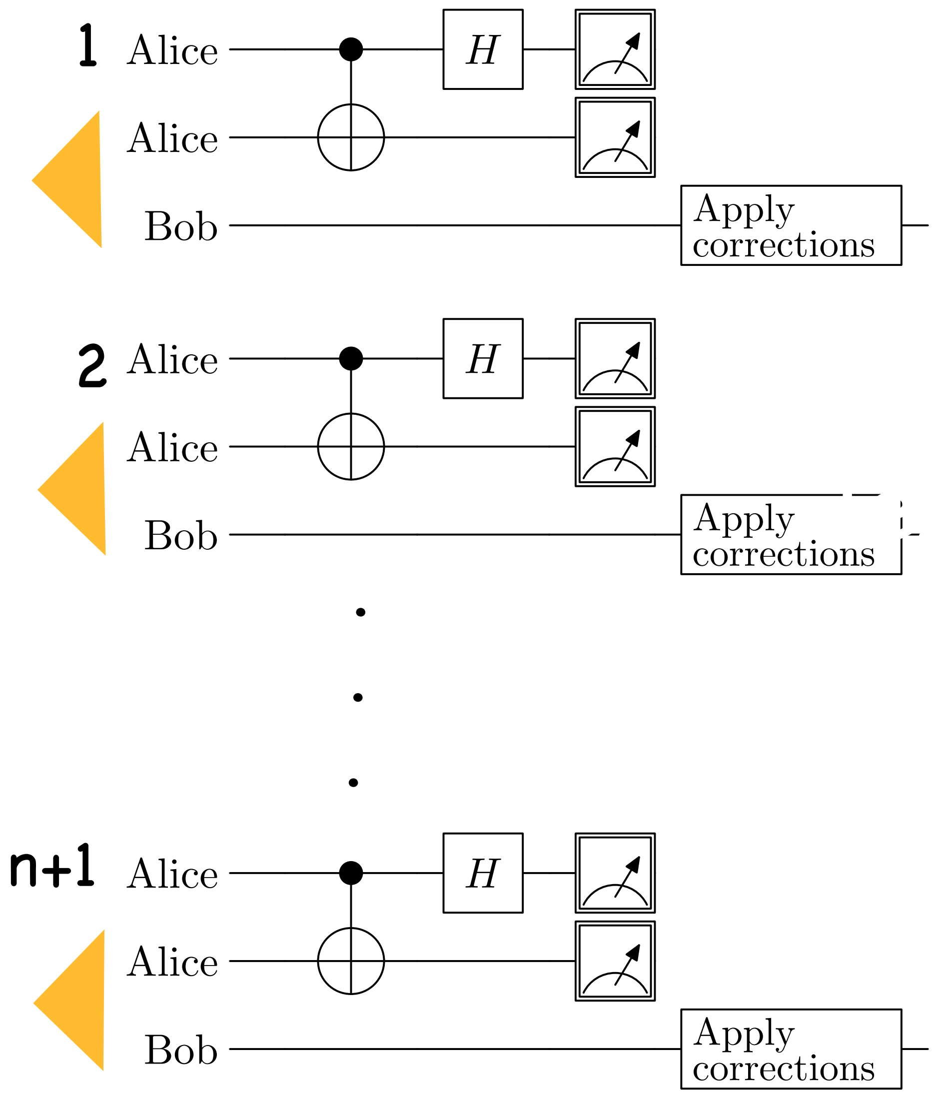
The Problem

Alice wants to send n+1 quantum bits to Bob. These quantum bits can be in any state.

We assume that they share n+1 pairs of entangled quantum bits in the state $(|00\rangle+|11\rangle)/\sqrt{2}$.

Can they solve the problem by separately teleporting each quantum bit?

The Quantum Circuit



n+1 quantum bits to be teleported
n+1 entangled pairs (triangles)

Is entanglement among n+1 quantum bits a problem?

Can we teleport one qubit at a time?

Initial State

We have n+1 quantum bits and want to teleport the least significant bit. Their state is:

$$\sum_{k=0}^{2^n-1} \sum_{j=0}^{1} a_{kj} |k\rangle \otimes |j\rangle \in \mathbf{C}^{2^n} \otimes \mathbf{C}^2$$

Goal: Show that after teleporting one quantum bit, the state of Bob's qubit and the remaining n qubits on Alice's side are in the same state, not entangled with anything else.

Applying Controlled Not

$$\sum_{k=0}^{2^n-1}\sum_{j=0}^{1} a_{kj}|k\rangle \otimes |j\rangle \otimes (\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle).$$

Apply $\text{CNOT}_{2,1}$:

$$\sum_{k=0}^{2^n-1}\Big(a_{k0}|k\rangle \otimes |0\rangle \otimes (\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle)$$
$$+a_{k1}|k\rangle \otimes |1\rangle \otimes (\frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|01\rangle)\Big).$$

Applying Hadamard

$$\sum_{k=0}^{2^n-1}\Big(a_{k0}|k\rangle \otimes |0\rangle \otimes (\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle)$$
$$+a_{k1}|k\rangle \otimes |1\rangle \otimes (\frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|01\rangle)\Big).$$

Apply Hadamard gate on position 2:

$$\sum_{k=0}^{2^n-1}\Big(a_{k0}|k\rangle \otimes \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|00\rangle + |11\rangle)$$
$$+a_{k1}|j\rangle \otimes \frac{1}{2}(|0\rangle - |1\rangle) \otimes (|10\rangle + |01\rangle)\Big).$$

Rewriting State

$$\sum_{k=0}^{2^n-1}\Big(a_{k0}|k\rangle \otimes \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|00\rangle + |11\rangle)$$

$$+a_{k1}|k\rangle \otimes \frac{1}{2}(|0\rangle - |1\rangle) \otimes (|10\rangle + |01\rangle)\Big)$$

$$= \sum_{k=0}^{2^n-1}\frac{1}{2}\Big(|k\rangle \otimes |00\rangle \otimes (a_{k0}|0\rangle + a_{k1}|1\rangle)$$

$$+|k\rangle \otimes |01\rangle \otimes (a_{k0}|1\rangle + a_{k1}|0\rangle)$$

$$+|k\rangle \otimes |10\rangle \otimes (a_{k0}|0\rangle - a_{k1}|1\rangle)$$

$$+|k\rangle \otimes |11\rangle \otimes (a_{k0}|1\rangle - a_{k1}|0\rangle)\Big)$$

Measurement and Correction

$$\sum_{k=0}^{2^n-1} \frac{1}{2}\Big(|k\rangle \otimes |00\rangle \otimes (a_{k0}|0\rangle + a_{k1}|1\rangle)$$
$$+|k\rangle \otimes |01\rangle \otimes (a_{k0}|1\rangle + a_{k1}|0\rangle)$$
$$+|k\rangle \otimes |10\rangle \otimes (a_{k0}|0\rangle - a_{k1}|1\rangle)$$
$$+|k\rangle \otimes |11\rangle \otimes (a_{k0}|1\rangle - a_{k1}|0\rangle)\Big).$$

Suppose that Alice measures the qubits at positions 2 and 1. If she observes x_2 and x_1, respectively, and informs Bob to apply $Z^{x_2}X^{x_1}$, then after applying Bob's correction operations, we get

$$\sum_{k=0}^{2^n-1}\sum_{j=0}^{1} |k\rangle \otimes |x_2x_1\rangle \otimes a_{kj}|j\rangle = \sum_{k=0}^{2^n-1}\sum_{j=0}^{1} a_{kj}|k\rangle \otimes |x_2x_1\rangle \otimes |j\rangle.$$

Conclusion

If Alice wants to communicate n+1 quantum bits to Bob, then she can do that by teleporting one quantum bit at a time.

How could you arrive at the same conclusion without any calculation at all?