Problem Set 4

Due dates: Electronic submission of the .pdf file of this homework is due on **9/27/2017 before 2:50pm** on ecampus, a signed paper copy of the pdf file is due on **9/27/2017** at the beginning of class.

Name: (put your name here)

Resources. (All people, books, articles, web pages, etc. that have been consulted when producing your answers to this homework)

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework.

Signature: ____________________________

Make sure that you describe all solutions in your own words. Read Chapter 16.4 and skim rest of the chapter.

Problem 1.

P 1. (10 points) Suppose that a country adopts coins of values 1, 9, and 21. Does the greedy algorithm to give change always give the fewest number of coins? Prove it or give the smallest counter example. Do not use the next problem.

Solution.

P 2. (20 points) Suppose that a coin system has values $1, c_2$, and c_3 with $1 < c_2 < c_3$. We say that the coin system is non-canonical if and only if there exists an amount $x > 1$ for which the greedy algorithm for giving change does not produce the fewest number of coins (i.e., the greedy algorithm is not optimal). Show that $(1, c_2, c_3)$ is non-canonical if and only if the quotient q and remainder r of dividing c_3 by c_2, that is, $c_3 = qc_2 + r$ with $0 \leq r < c_2$, satisfy $0 < r < c_2 - q$.
[Hint: You can freely use the fact proven by Kozen and Zaks that the smallest counterexample x is in the range $c_3 + 1 < x < c_2 + c_3$. Show that the smallest counterexample x uses just coins of value c_2 in the optimal solution, and only coins of value 1 and c_3 in the greedy solution.]

Solution.

P 3. (10 points) Consider Kruskals algorithm for a hypercube graph $Q_3 = (V, E)$ with vertices $V = \{000, 001, \ldots, 111\}$. We (u, v) in E if and only if the Hamming distance betweeen u and v is 1. Put differently, two vertices are connected by an edge if and only if the labels of the vertices differ in a single bit. The weight of an edge (u, v) is given by converting the 6 bit concatenated bit string uv into decimal, where we always order the edge such that the decimal number satisfy $u < v$. In which order will Kruskal's algorithm pick the edges?

Solution.

P 4. (20 points) The specification of a matroid $M = (S, F)$ as a finite set S together with a nonempty family F of subsets of S that is hereditary and satisfies the exchange axiom is not particularly economical. A set in F that is maximal with respect to inclusion is called a basis of the matroid. Let $\mathcal{B}$ denote the set of bases of M. Then $(S, \mathcal{B})$ is a more economical way to represent the matroid M. Show that

(a) $\mathcal{B}$ is not empty,
(b) all bases in $\mathcal{B}$ have the same cardinality,
(c) the set family $\mathcal{B}$ satisfies the condition: If B_1 and B_2 are bases in $\mathcal{B}$ and $x \in B_1 \backslash B_2$, then there exists an element $y \in B_2 \backslash B_1$ such that $(B_1 \backslash \{x\}) \cup \{y\}$ is a basis in $\mathcal{B}$.
(d) one can recover the matroid (S, F) from $(S, \mathcal{B})$,

(e) more generally, given a finite set S and a family $\mathcal{B}$ of subsets of S satisfying (a), (b), and (c) gives rise to a matroid using your technique from (d).

Solution.

P 5. (20 points) Exercise 16.4-4 on page 443 in our textbook.

Solution.

P 6. (20 points) Let S be a finite set, F a nonempty family of subsets of S that satisfies the hereditary axiom. Show that if (S, F) is **not** a matroid, that is, does not satisfy the exchange axiom, then there exists a weight function $w\colon S \to \mathbf{R}_{\geq 0}$ such that Greedy((S, F), s) does not return a maximum weight basis of F, (a basis is a set in F that is not contained in any larger set in F). [Hint: Consider two subsets A and B such that $|A| < |B|$ but such that there does not exist any $x \in B \setminus A$ satisfying $A \cup \{x\}$ in F. Assume that A has m elements and construct a weight w such that the algorithm will return a set that has weight $w(A)$ even though $w(A) < w(B)$.]

Solution.

Discussions on piazza are always encouraged, especially to clarify concepts that were introduced in the lecture. However, discussions of homework problems on piazza should not contain spoilers. It is okay to ask for clarifications concerning homework questions if needed.

Checklist:
- ☐ Did you add your name?
- ☐ Did you disclose all resources that you have used?
  (This includes all people, books, websites, etc. that you have consulted)
- ☐ Did you sign that you followed the Aggie honor code?
- ☐ Did you solve all problems?
- ☐ Did you submit the pdf (derived from your latex file) of your homework?
- ☐ Did you submit a hardcopy of the pdf file in class?