Analysis of Algorithms

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Motivation

In 2004, a mysterious billboard showed up

- in the Silicon Valley, CA
- in Cambridge, MA
- in Seattle, WA
- in Austin, TX

and perhaps a few other places.

Remarkably, the puzzle on the billboard was immediately discussed worldwide in numerous blogs.

Motivation

{first 10-digit prime found in consecutive digits of e}.com
CLEARCHANNEL

Recall Euler's Number e

$$e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$$

$$\approx 2.7182818284\ldots$$

Billboard Question

So the billboard question essentially asked: Given that

$$e = 2.7182818284\ldots$$

$$\text{Is } 2718281828 \text{ prime?}$$

$$\text{Is } 7182818284 \text{ prime?}$$

The first affirmative answer gives the name of the website

Strategy

1. Compute the digits of e
2. i := 0
3. while true do {
4. Extract 10 digit number p at position i
5. return p if p is prime
6. i := i+1
7. }

What needs to be solved?

Essentially, two questions need to be solved:

- How can we create the digits of e?
- How can we test whether an integer is prime?

Generating the Digits

Extracting Digits of e

We can extract the digits of e in base 10 by

d[0] = floor(e); (equals 2)

e1 = 10*(e-d[0]);

d[1] = floor(e1); (equals 7)

e2 = 10*(e1-d[1]);

d[2] = floor(e2); (equals 1)

Unfortunately, e is a transcendental number, so there is **no pattern** to the generation of the digits in base 10.

Initial idea: Use rational approximation to e instead

Some Bounds on e=exp(1)

For any t in the range $1 \le t \le 1 + 1/n$, we have

$$\frac{1}{1+\frac{1}{n}} \le \frac{1}{t} \le 1.$$

Hence,

$$\int_1^{1+1/n} \frac{1}{1+\frac{1}{n}}\,dt \le \int_1^{1+1/n} \frac{1}{t}\,dt \le \int_1^{1+1/n} 1\,dt.$$

Thus,

$$\frac{1}{n+1} \le \ln\left(1+\frac{1}{n}\right) \le \frac{1}{n}$$

Exponentiating

$$\frac{1}{n+1} \leq \ln\left(1+\frac{1}{n}\right) \leq \frac{1}{n}$$

yields

$$e^{1/n+1} \leq \left(1+\frac{1}{n}\right) \leq e^{\frac{1}{n}}.$$

Therefore, we can conclude that

$$\left(1+\frac{1}{n}\right)^{n} \leq e \leq \left(1+\frac{1}{n}\right)^{n+1}.$$

Approximating e

Since

$$\left(1+\frac{1}{n}\right)^n \leq e \leq \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right),$$

the term

$$\left(1+\frac{1}{n}\right)^n$$

approximates e to k digits, when choosing $n = 10^{k+1}$.

Drawbacks

- The rational approximation converges too slow.
- We need rational arithmetic with long rationals
- Too much coding unless a library is used.
- Perhaps we can find a better solution by choosing a better data structure.

Generating the Digits
Version 2

Idea

- e is a transcendental number => no pattern when generating its digits in the usual number representation
- Can we find a better data structure?

Mixed Radix Representation

$$a_0 + \frac{1}{2}\left(a_1 + \frac{1}{3}\left(a_2 + \frac{1}{4}\left(a_3 + \frac{1}{5}\left(a_4 + \frac{1}{6}\left(a_5 + \cdots\right)\right)\right)\right)\right)$$

The digits a_i are nonnegative integers.

The base of this representation is (1/2,1/3,1/4,...).

The representation is called regular if

$a_i <= i$ for $i >=1$.

Number is written as $(a_{0;}\ a_{1,}\ a_{2,}\ a_{3,}...)$

Computing the Digits of the Number e

- Second approach:

$$\begin{aligned} e &= \sum_{k=0}^{\infty} \frac{1}{k!} \\ &= 1 + \tfrac{1}{1}\left(1 + \tfrac{1}{2}\left(1 + \tfrac{1}{3}\left(1 + \cdots\right)\right)\right) \end{aligned}$$

- In mixed radix representation e = (2;1,1,1,1,...) where the digit 2 is due to the fact that both k=0 and k=1 contribute to the integral part. Remember: 0!=1 and 1!=1.

Mixed Radix Representations

- In mixed radix representation **($a_{0;}$ $a_{1,}$ $a_{2,}$ $a_{3,}$...)**

  a_0 is the integer part and **($0_{;}$ $a_{1,}$ $a_{2,}$ $a_{3,}$...)** the fractional part.

- 10 times the number is **($10a_{0;}$ $10a_{1,}$ $10a_{2,}$ $10a_{3,}$...)**, but the representation is not regular anymore. The first few digits might exceed their bound. Remember that the ith digit is supposed to be i or less.

- Renormalize the representation to make it regular again

- The algorithm given for base 10 now becomes feasible; this is known as the spigot algorithm.

$$e = 2 + \left[\frac{1}{2}\left(1 + \frac{1}{3}\left(1 + \frac{1}{4}\left(1 + \frac{1}{5}\right.\right.\right.\right.$$

$$\left.\left.\left.\left(1 + \frac{1}{6}\left(1 + \dots \right)\right)\right)\right)$$

$$= 2 + \frac{1}{10}\left[\frac{1}{2}\left(10 + \frac{1}{3}\left(10 + \frac{1}{4}\left(10 + \frac{1}{5}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left(10 + \frac{1}{6}\left(10 + \dots \right)\right)\right)\right)\right)\right]$$

$$= 2 + \frac{1}{10}\left[7 + \frac{1}{2}\left(0 + \frac{1}{3}\left(1 + \frac{1}{4}\left(0 + \frac{1}{5}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left(1 + \frac{1}{6}\left(5 + \dots \right)\right)\right)\right)\right)\right]$$

$$
\begin{aligned}
&= 2\cdot 7 + \frac{1}{100}\Big[\frac{1}{2}\Big(0 + \frac{1}{3}\Big(10 + \frac{1}{4}\Big(0 + \frac{1}{5} \\
&\qquad\qquad \Big(10 + \frac{1}{6}\Big(50 + \ldots \quad \Big)\Big)\Big)\Big)\Big)\Big] \\
&= 2\cdot 7 + \frac{1}{100}\Big[1 + \frac{1}{2}\Big(1 + \frac{1}{3}\Big(1 + \frac{1}{4}\Big(3 + \frac{1}{5} \\
&\qquad\qquad \Big(4 + \frac{1}{6}\Big(2 + \ldots \quad \Big)\Big)\Big)\Big)\Big)\Big]
\end{aligned}
$$

Spigot Algorithm

```
#define N (1000) /* compute N-1 digits of e, by brainwagon@gmail.com */
main( i, j, q ) {
   int A[N]; printf("2.");
   for ( j = 0; j < N; j++ )
      A[j] = 1;                    here the ith digit is represented by A[i-1], as the integral part is omitted
                                   set all digits of nonintegral part to 1.
   for ( i = 0; i < N - 2; i++ ) {
      q = 0;
      for ( j = N - 1; j >= 0; ) {
         A[j] = 10 * A[j] + q;
         q = A[j] / (j + 2);       compute the amount that needs to be carried over to the next digit
                                   we divide by j+2, as regularity means here that A[j] <= j+1
         A[j] %= (j + 2);          keep only the remainder so that the digit is regular
         j--;
      }
      putchar(q + 48);
   }
}
```

Revisiting the Question

For mathematicians, the previous algorithm is natural, but it might be a challenge for computer scientists and computer engineers to come up with such a solution.

Could we get away with a simpler approach?

After all, the billboard only asks for the **first** prime in the 10-digit numbers occurring in e.

Generating the Digits
Version 3

Probability to be Prime

Let pi(x)=# of primes less than or equal to x.

Pr[number with <= 10 digits is prime]

= pi(99999 99999)/99999 99999

= 0.045 (roughly)

Thus, the probability that **none** of the first k 10-digits numbers in e are prime is roughly 0.955^k

This probability rapidly approaches 0 for k->∞, so we need to compute just a few digits of e to find the first 10-digit prime number in e.

Google it!

Since we will likely need just few digits of Euler's number e, there is no need to reinvent the wheel.

We can simply

- google e or
- use the GNU bc calculator

to obtain a few hundred digits of e.

State of Affairs

We have provided three solutions to the question of generating the digits of e

- A straightforward solution using rational approximation

- An elegant solution using the mixed-radix representation of e that led to the spigot algorithm

- A crafty solution that provides enough digits of e to solve the problem at hand.

How do we check Primality?

The second question concerning the testing of primality is simpler.

If a number x is not prime, then it has a divisor d in the range 2<= d <= sqrt(x).

Trial divisions are fast enough here!

Simply check whether any number d in the range 2 <= d < 100 000 divides a 10-digit chunk of e.

A Simple Script

http://discuss.fogcreek.com/joelonsoftware/default.asp?cmd=show&ixPost=160966&ixReplies=23

```
#!/bin/sh
echo "scale=1000; e(1)" | bc -l | \
perl -0777 -ne '
s/[^0-9]//g;
for $i (0..length($_)-10) {
  $j=substr($_,$i,10);
  $j +=0;
  print "$i\t$j\n" if is_p($j);
}
```

```
sub is_p {
  my $n = shift;
  return 0 if $n <= 1;
  return 1 if $n <= 3;
  for (2 .. sqrt($n)) {
    return 0 unless $n % $_;
  }
 return 1;
```

What was it all about?

The billboard was an ad paid for by Google. The website

http://www.7427466391.com

contained another challenge and then asked people to submit their resume.

Google's obsession with e is well-known, since they pledged in their IPO filing to raise e billion dollars, rather than the usual round-number amount of money.

Summary

- Rational approximation to e and primality test by trial division
- Spigot algorithm for e and primality test by trial division
- A simple crafty solution