The Bellman-Ford Algorithm

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Single Source Shortest Path Problem

Given a graph G=(V,E), a weight function w: E -> R, and a source node s, find the shortest path from s to v for every v in V.

- We allow negative edge weights.
- G is not allowed to contain cycles of negative total weight.
- Dijkstra's algorithm cannot be used, as weights must be nonnegative.

Bellman-Ford SSSP Algorithm

Input: directed or undirected graph G = (V,E,w)

```
for all v in V {
   d[v] = infinity; parent[v] = nil;
}
d[s] = 0; parent[s] = s;
for i := 1 to |V| - 1 { // ensure that information on distance from s propagates
    for each (u,v) in E { // relax all edges
       if (d[u] + w(u,v) < d[v]) then {  d[v] := d[u] + w(u,v); parent[v] := u; }
    }
}
```

Running Time: O(VE)

```
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    }
}
```
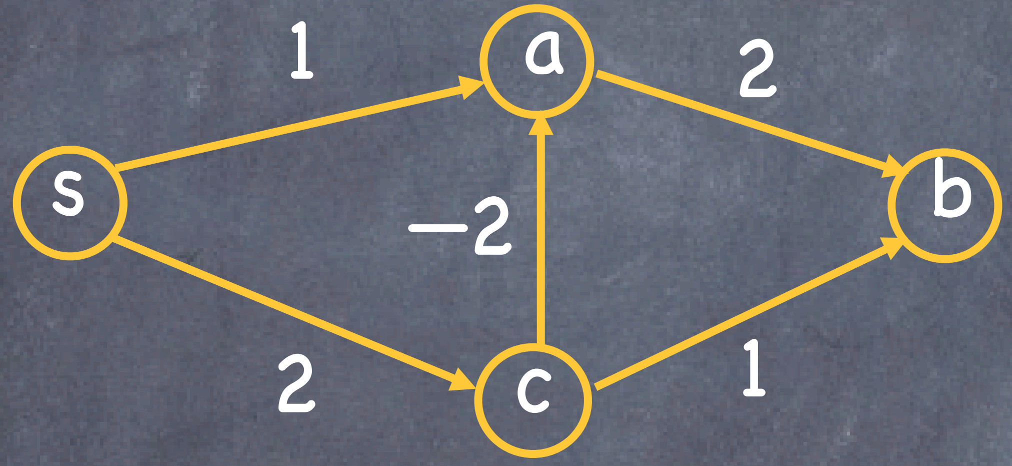

Init: $O(V)$

Nested loops: $O(V)O(E)=O(VE)$

Bellman-Ford Example



Let's process edges in the order
(c,b),(a,b),(c,a),(s,a),(s,c)

	Iteration			
Node	0	1	2	3
s	0	0	0	0
a	∞	1	0	0
b	∞	∞	3	2
c	∞	2	2	2

Information Propagation

Consider a graph on n+1 vertices:

$s \to a_1 \to a_2 \to \ldots \to a_{n-1} \to a_n$

where each edge has weight 1.

Choose edges from right to left. first. Then node a_i has correct distance estimate after i^{th} iteration.

	Iteration				
Node	0	1	2	3	4
s	0	0	0	0	
a	∞	1	1	1	...
a	∞	∞	2	2	...
a	∞	∞	∞	3	...
a	∞	∞	∞	∞	...

Correctness

Fact 1: The distance estimate d[v] never underestimates the actual shortest path distance from s to v.

Fact 2: If there is a shortest path from s to v containing at most i edges, then after iteration i of the outer for loop:

d[v] <= the actual shortest path distance from s to v.

Correctness

Theorem: Suppose that G is a weighted graph without negative weight cycles and let s denote the source node. Then Bellman-Ford correctly calculates the shortest path distances from s.

Proof: Every shortest path has at most $|V| - 1$ edges. By Fact 1 and 2, the distance estimate $d[v]$ is equal to the shortest path length after $|V|-1$ iterations.

Variations

One can stop the algorithm if an iteration does not modify distance estimates. This is beneficial if shortest paths are likely to be less than |V|-1.

One can detect negative weight cycles by checking whether distance estimates can be reduced after |V|-1 iterations.

The Boost Graph Library

The BGL contains generic implementations of all the graph algorithms that we have discussed:

- Breadth-First-Search
- Depth-First-Search
- Kruskal's MST algorithm
- Strongly Connected Components
- Dijkstra's SSSP algorithm
- Bellman-Ford SSSP algorithm

I recommend that you gain experience with this useful library. Recommended reading: The Boost Graph Library by J.G. Siek, L.-Q. Lee, and A. Lumsdaine, Addison-Wesley, 2002.