How Many Repetitions?

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Suppose that we have a randomized Monte Carlo algorithm that succeeds with probability

$$\frac{1}{p(n)},$$

where $p(n)$ is some polynomial in the length of the input.

Question

How many times do we need to repeat the algorithm to get success with high probability (meaning with probability approaching 1)?

Useful Inequality

$$1 + x \leqslant e^x.$$

Consider the function $f(x) = e^x - 1 - x$. Our inequality is equivalent to $f(x) \geqslant 0$, so our goal will be to prove that.

The function $f(x)$ has the derivative $f'(x) = e^x - 1$.

We have $f'(x) = 0$ if and only if $x = 0$.

The function $f(x)$ has a (global) minimum at $x = 0$, since $f''(0) = e^0 = 1 > 0$. We can conclude that $f(x) \geqslant f(0) = 0$.

Corollary

Consequently, for all positive integers n, we have

$$\left(1+\frac{x}{n}\right)^n \leqslant (e^{x/n})^n = e^x.$$

Corollary

Let $p(n)$ be a polynomial in n. For all positive integers n, we have

$$\left(1-\frac{1}{p(n)}\right)^{p(n)} \leqslant (e^{-1/p(n)})^{p(n)} = e^{-1}.$$

Corollary

Let $p(n)$ be a polynomial in n. For all positive integers n, we have

$$\left(1 - \frac{1}{p(n)}\right)^{p(n)\ln n} \leqslant (e^{-1/p(n)})^{p(n)\ln n} = e^{-\ln n} = \frac{1}{n}.$$

Conclusion

Thus, if we repeat an algorithm that succeeds with probability $1/p(n)$ for $p(n)\ln n$ times, then we get with high probability a correct result.

Conclusion

If a Monte Carlo algorithm gives the correct result for inputs of size n with probability at least $1/p(n)$, and we repeat the algorithm

$$p(n) \ln n$$

times, then probability that it will never show the correct result in any of these repeated runs is at most $1/n$.

In other words, we get the correct result with probability $1 - 1/n$.