Sorting Lower Bound

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[based on slides by Prof. Welch]

Warm Up: Review of a Few Sorting Algorithms

Insertion Sort Review

How it works:

- incrementally build up longer and longer prefix of the array of keys that is in sorted order
- take the current key, find correct place in sorted prefix, and shift to make room to insert it

Finding the correct place relies on *comparing* current key to keys in sorted prefix

Worst-case running time is $\Theta(n^2)$

Insertion Sort Demo

Insertion Sort Animation

Insertion Sort

```
// Sort A[0..N-1]
i ← 1
while i < length(A)
    j ← i
    while j > 0 and A[j-1] > A[j]
        swap A[j] and A[j-1]
        j ← j - 1
    end while
    i ← i + 1
end while
```

Heapsort Review

How it works:

- put the keys in a heap data structure
- repeatedly remove the max from the heap

Manipulating the heap involves comparing keys to each other

Worst-case running time is $\Theta(n \log n)$

Heapsort Demo

Heapsort Animation Video

[Note that aces are high]

Mergesort Review

How it works:

- split the array of keys in half
- recursively sort the two halves
- merge the two sorted halves

Merging the two sorted halves involves *comparing* keys to each other

Worst-case running time is $\Theta(n \log n)$

Mergesort Demo

Mergesort Animation

Quicksort Review

How it works:

- choose one key to be the *pivot*
- partition the array of keys into

  those keys < the pivot and those ≥ the pivot
- recursively sort the two partitions

Partitioning the array involves *comparing* keys to the pivot. Worst-case running time is $\Theta(n^2)$

Quicksort Demo

Quicksort Video

[You can mute the video]

Comparison Based Sorting

Comparison-Based Sorting

All these algorithms are *comparison-based*

- the behavior depends on relative values of keys, not exact values
- behavior on [1,3,2,4] is same as on [9,25,23,99]

Fastest of these algorithms was $O(n \log n)$.

We will show that's the best you can get with comparison-based sorting.

Lower Bounds

Decision Tree

Consider *any* comparison based sorting algorithm

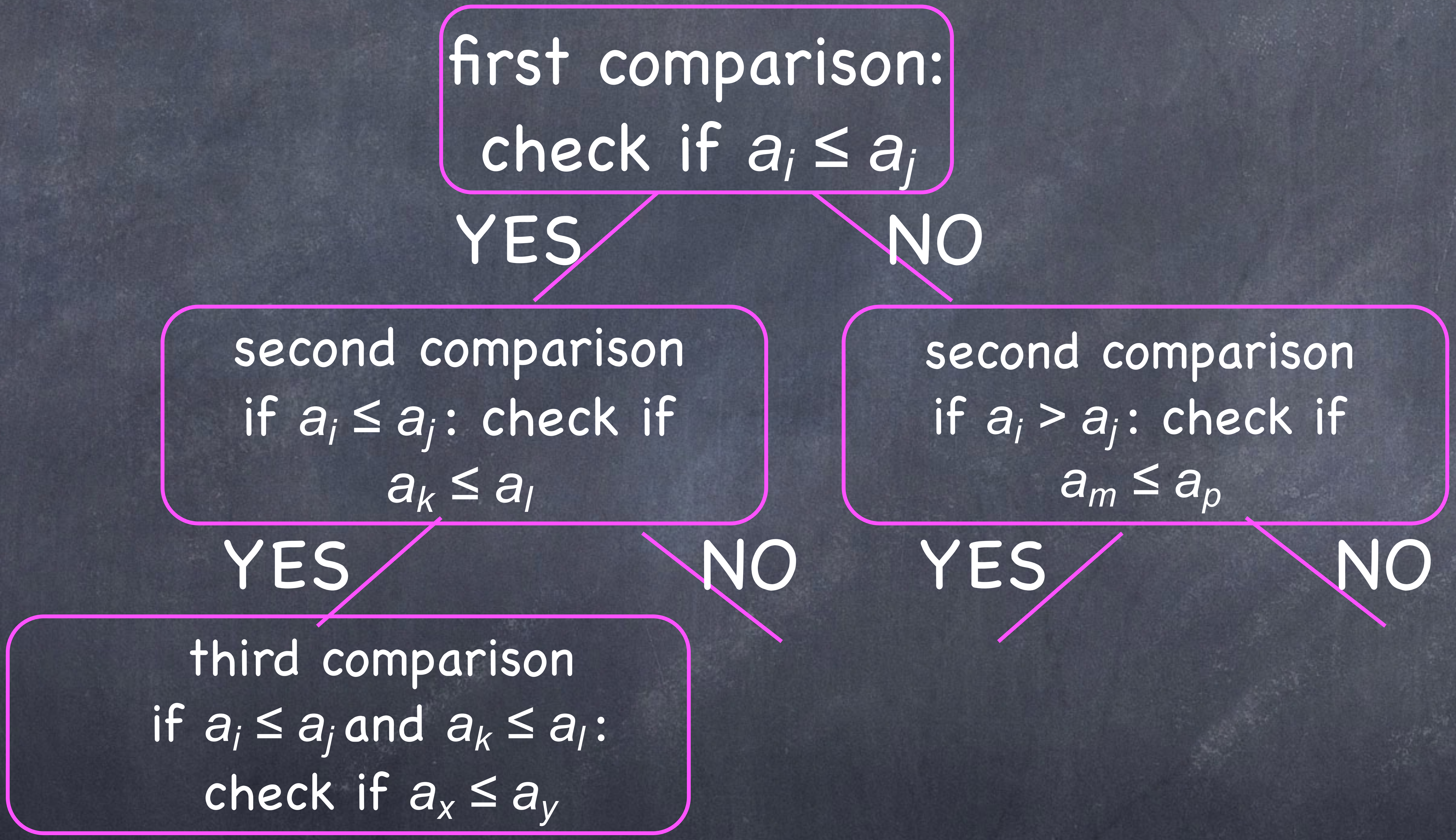
Represent its behavior on all inputs of a fixed size with a *decision tree*

Each tree node corresponds to the execution of a comparison

Each tree node has two children, depending on whether the parent comparison was true or false

Each leaf represents correct sorted order for that path

Decision Tree Diagram

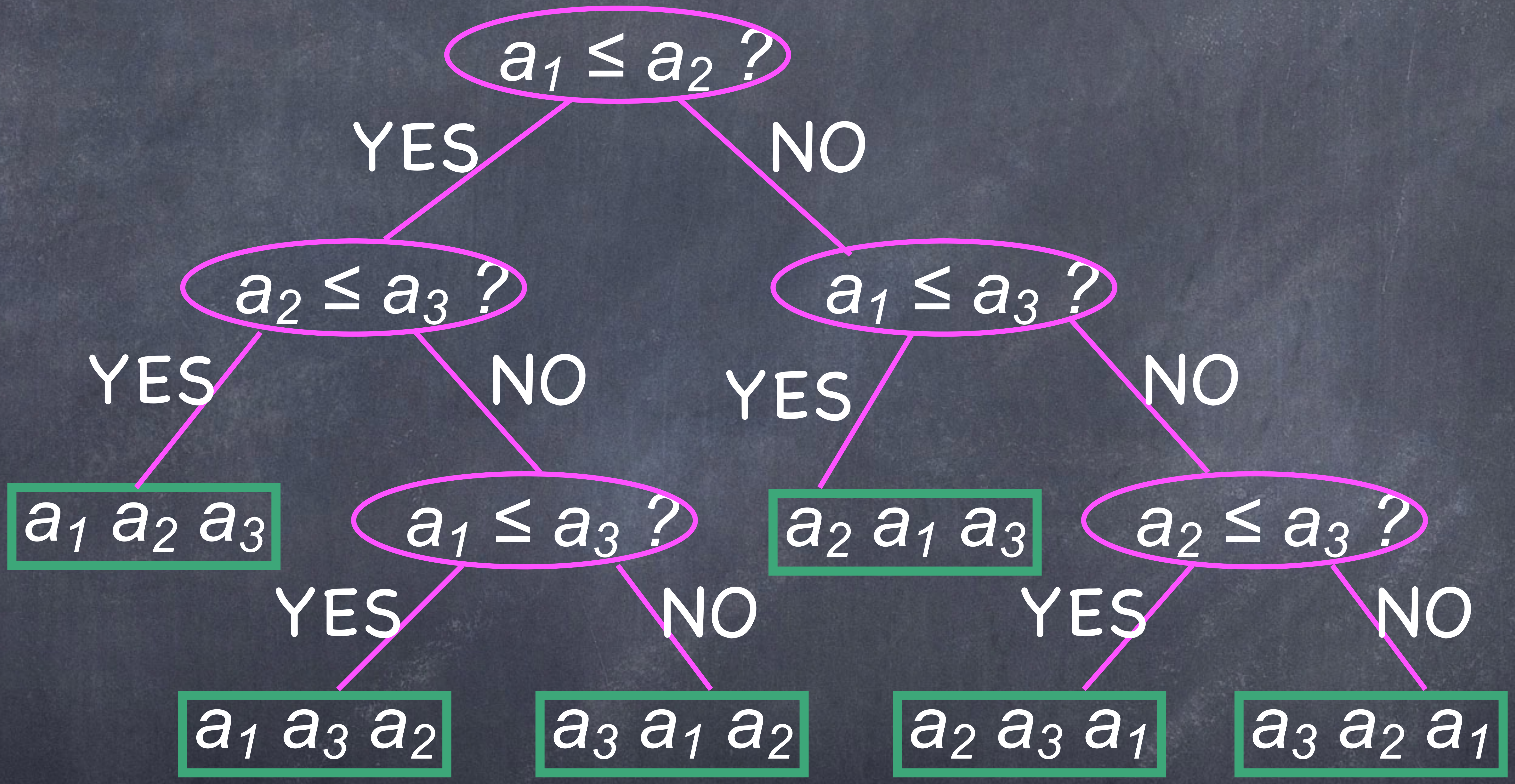


Insertion Sort

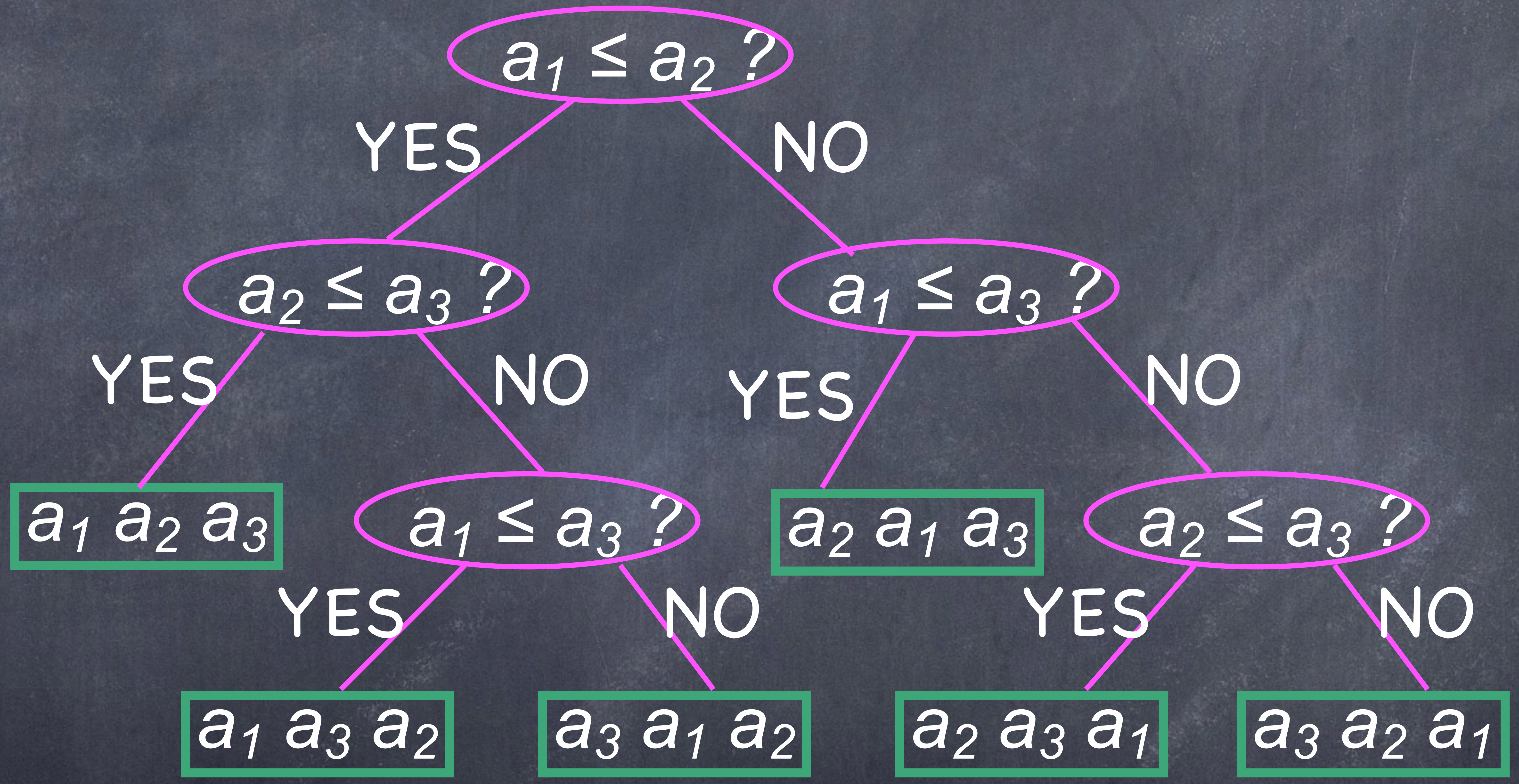
comparison

```
for j := 2 to n to
   key := a[j]
   i := j-1
   while i > 0 and a[i] > key do // insert in prev.
      a[i+1] := a[i]
      i := i -1
   endwhile
   a[i+1] := key
endfor
```

Insertion Sort for $n = 3$



Insertion Sort for $n = 3$



How Many Leaves?

- Must be at least one leaf for each permutation of the input
  - otherwise there would be a situation that was not correctly sorted
- Number of permutations of n keys is $n!$.
- Idea: since there must be a lot of leaves, but each decision tree node only has two children, tree cannot be too shallow
  - depth of tree is a lower bound on running time

Key Lemma

Height of a binary tree with *n!* leaves is

$\Omega(n \log n)$.

Proof: The maximum number of leaves in a binary tree with height h is 2^h.

$h = 1$, 2^1 leaves

$h = 2$, 2^2 leaves

$h = 3$, 2^3 leaves

Proof of Lemma

- Let h be the height of decision tree, so it has at most 2^h leaves.
- The actual number of leaves is $n!$, hence

$2^h \geq n!$

$h \geq \log(n!)$

$= \log(\ n(n-1)(n-2)\ \ldots\ (2)(1)\ )$

$\geq (n/2)\log(n/2)$ [Why?]

$= \Omega(n \log n)$

Finishing Up

- Any binary tree with $n!$ leaves has height $\Omega(n \log n)$.
- Decision tree for any c-b sorting alg on n keys has height $\Omega(n \log n)$.
- Any c-b sorting alg has at least one execution with $\Omega(n \log n)$ comparisons
- Any c-b sorting alg has $\Omega(n \log n)$ worst-case running time.