Dynamic Programming: The Matrix Chain Algorithm

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[partially based on slides by Prof. Welch]

Matrix Chain Problem

Suppose that we want to multiply a sequence of rectangular matrices. In which order should we multiply?

$$A \times (B \times C) \quad \text{or} \quad (A \times B) \times C$$

Matrices

An n x m matrix A over the real numbers is a rectangular array of nm real numbers that are arranged in n rows and m columns.

For example, a 3 x 2 matrix A has 6 entries

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

where each of the entries a_{ij} is e.g. a real number.

Matrix Multiplication

Let A be an n x m matrix

B an m x p matrix

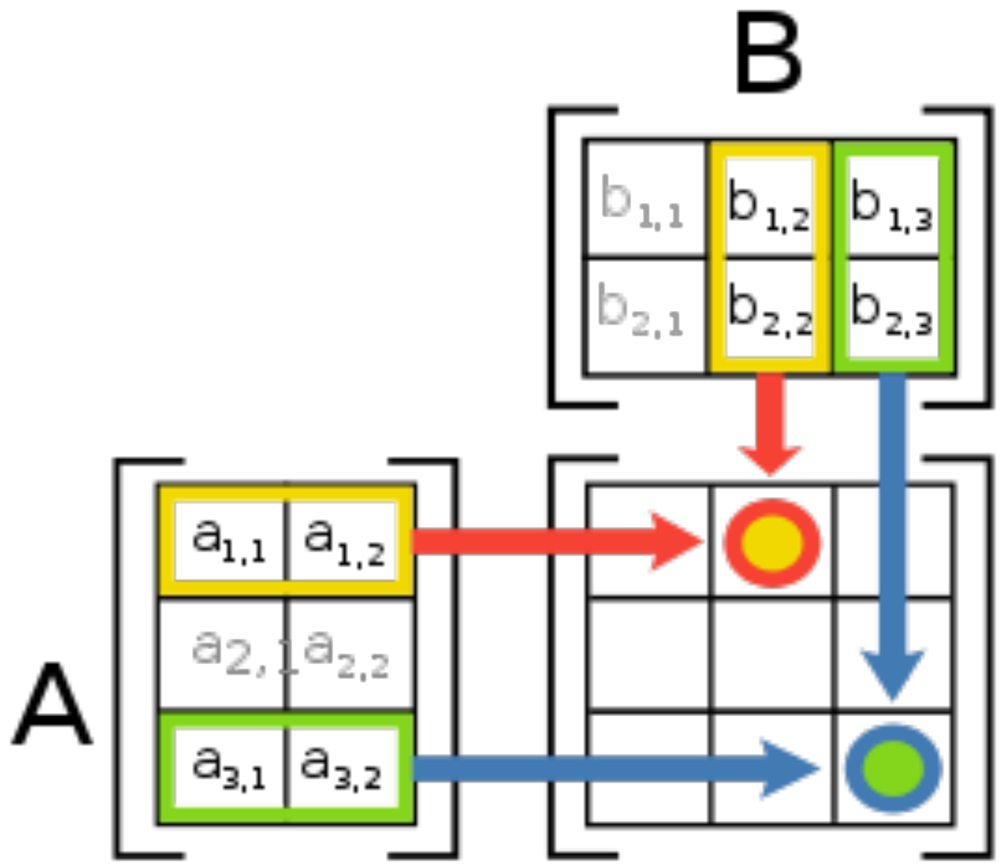
The product of A and B is n x p matrix AB whose (i,j)-th entry is

$\Sigma_{k=1}^{m} a_{ik} b_{kj}$

In other words, we multiply the entries of the i-th row of A with the entries of the j-th column of B and add them up.

Matrix Multiplication

$$
\begin{aligned}
\color{red}{x_{1,2}} &= (a_{1,1}, a_{1,2}) \cdot (b_{1,2}, b_{2,2}) \\
&= a_{1,1}b_{1,2} + a_{1,2}b_{2,2} \\
\color{blue}{x_{3,3}} &= (a_{3,1}, a_{3,2}) \cdot (b_{1,3}, b_{2,3}) \\
&= a_{3,1}b_{1,3} + a_{3,2}b_{2,3}.
\end{aligned}
$$



Complexity of Matrix Multiplication

Let A be an $n \times m$ matrix, B an $m \times p$ matrix. Thus,

AB is an $n \times p$ matrix. Computing the product AB takes

nmp scalar multiplications

$n(m-1)p$ scalar additions

for the standard matrix multiplication algorithm.

Matrix Chain Order Problem

Matrix multiplication is associative, meaning that (AB)C = A(BC). Therefore, we have a choice in forming the product of several matrices.

What is the least expensive way to form the product of several matrices if the naïve matrix multiplication algorithm is used?

[We use the number of scalar multiplications as cost.]

Why Order Matters

Suppose we have 4 matrices:

A: 30 x 1

B: 1 x 40

C: 40 x 10

D: 10 x 25

((AB)(CD)) : requires 41,200 scalar multiplications

(A((BC)D)) : requires 1400 scalar multiplications

Matrix Chain Order Problem

Given matrices $A_1, A_2, \ldots, A_n$,

where A_i is a $d_{i-1} \times d_i$ matrix.

[1] What is minimum number of scalar multiplications required to compute the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$?

[2] What order of matrix multiplications achieves this minimum?

We focus on question [1], and sketch an answer to [2].

A Possible Solution

Try all possibilities and choose the best one.

Drawback: There are too many of them (exponential in the number of matrices to be multiplied)

We need to be smarter: Let’s try dynamic programming!

Step 1: Develop a Recursive Solution

- Define $M(i,j)$ to be the minimum number of multiplications needed to compute $A_i \cdot A_{i+1} \cdot \ldots \cdot A_j$
- Goal: Find $M(1,n)$.
- Basis: $M(i,i) = 0$.
- Recursion: How can one define $M(i,j)$ recursively?

Defining $M(i,j)$ Recursively

- Consider all possible ways to split A_i through A_j into two pieces.
- Compare the costs of all these splits:
  - best case cost for computing the product of the two pieces
  - plus the cost of multiplying the two products
- Take the best one
- $M(i,j) = \min_k(M(i,k) + M(k+1,j) + d_{i-1}d_kd_j)$

Defining M(i,j) Recursively

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{P_1} \cdot \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{P_2}$$

- minimum cost to compute P_1 is $M(i,k)$
- minimum cost to compute P_2 is $M(k+1,j)$
- cost to compute $P_1 \cdot P_2$ is $d_{i-1} d_k d_j$

Step 2: Find Dependencies Among Subproblems

M:

	1	2	3	4	5
1	0				
2	n/a	0			
3	n/a	n/a	0		
4	n/a	n/a	n/a	0	
5	n/a	n/a	n/a	n/a	0

GOAL!

computing the pink square requires the purple ones: to the left and below.

Defining the Dependencies

Computing $M(i,j)$ uses

everything in same row to the left:

$M(i,i), M(i,i+1), \ldots, M(i,j-1)$

and everything in same column below:

$M(i,j), M(i+1,j),\ldots,M(j,j)$

Step 3: Identify Order for Solving Subproblems

Recall the dependencies between subproblems just found

Solve the subproblems (i.e., fill in the table entries) this way:

- go along the diagonal
- start just above the main diagonal
- end in the upper right corner (goal)

Order for Solving Subproblems

M:

	1	2	3	4	5
1	0				
2	n/a	0			
3	n/a	n/a	0		
4	n/a	n/a	n/a	0	
5	n/a	n/a	n/a	n/a	0

Pseudocode

```
for i := 1 to n do M[i,i] := 0
for d := 1 to n-1 do  // diagonals
  for i := 1 to n-d to // rows w/ an entry on d-th diagonal
    j := i + d          // column corresp. to row i on d-th diagonal
    M[i,j] := infinity
    for k := i to j-1 to
      M[i,j] := min(M[i,j], M[i,k]+M[k+1,j]+d_{i-1}d_kd_j)
    endfor
  endfor
endfor
```

pay attention here to remember actual sequence of mults.

running time $O(n^3)$

Example

M:

	1	2	3	4
1	0	1200	700	1400
2	n/a	0	400	650
3	n/a	n/a	0	10,000
4	n/a	n/a	n/a	0

1: A is 30x1
2: B is 1x40
3: C is 40x10
4: D is 10x25

BxC: 1x40x10

(BxC)xD:
400 + 1x10x25

Bx(CxD):
... + 10,000

Keeping Track of the Order

- It's fine to know the cost of the cheapest order, but what is that cheapest order?
- Keep another array S and update it when computing the minimum cost in the inner loop
- After M and S have been filled in, then call a recursive algorithm on S to print out the actual order

Modified Pseudocode

```
for i := 1 to n do M[i,i] := 0
for d := 1 to n-1 do  // diagonals
  for i := 1 to n-d to // rows w/ an entry on d-th diagonal
    j := i + d          // column corresponding to row i on d-th diagonal
    M[i,j] := infinity
    for k := i to j-1 to
      M[i,j] := min(M[i,j], M[i,k]+M[k+1,j]+d_{i-1}d_kd_j)
      if previous line changed value of M[i,j] then S[i,j] := k
    endfor
  endfor
endfor
```

keep track of cheapest split point found so far: between A_k and A_{k+1}

Example

M:

S:

	1	2	3	4
1	0	1200 1	700 1	1400 1
2	n/a	0	400 2	650 3
3	n/a	n/a	0	10,000 3
4	n/a	n/a	n/a	0

1: A is 30x1
2: B is 1x40
3: C is 40x10
4: D is 10x25

A x (BCD)
A x ((BC) x D)
A x ((BxC) x D)

Using S to Print Best Ordering

Call Print(S,1,n) to get the entire ordering.

Print(S,i,*j*):

    if i = *j* then output "A" + i   //+ is string concat

    else

        k := S[i,*j*]

        output "(" + Print(S,i,k) + Print(S,k+1,*j*) + ")"