Sorting Lower Bound

Andreas Klappenecker
based on slides by Prof. Welch

Insertion Sort Review

- How it works:
  - incrementally build up longer and longer prefix of the array of keys that is in sorted order
  - take the current key, find correct place in sorted prefix, and shift to make room to insert it
- Finding the correct place relies on comparing current key to keys in sorted prefix
- Worst-case running time is $\Theta(n^2)$

Insertion Sort Demo

- http://sorting-algorithms.com

Heapsort Review

- How it works:
  - put the keys in a heap data structure
  - repeatedly remove the min from the heap
- Manipulating the heap involves comparing keys to each other
- Worst-case running time is $\Theta(n \log n)$

Heapsort Demo

- http://www.sorting-algorithms.com

Mergesort Review

- How it works:
  - split the array of keys in half
  - recursively sort the two halves
  - merge the two sorted halves
- Merging the two sorted halves involves comparing keys to each other
- Worst-case running time is $\Theta(n \log n)$

Mergesort Demo

- http://www.sorting-algorithms.com

Quicksort Review

- How it works:
  - choose one key to be the pivot
  - partition the array of keys into those keys $<$ the pivot and those $\geq$ the pivot
  - recursively sort the two partitions
- Partitioning the array involves comparing keys to the pivot
- Worst-case running time is $\Theta(n^2)$

Quicksort Demo

- http://www.sorting-algorithms.com

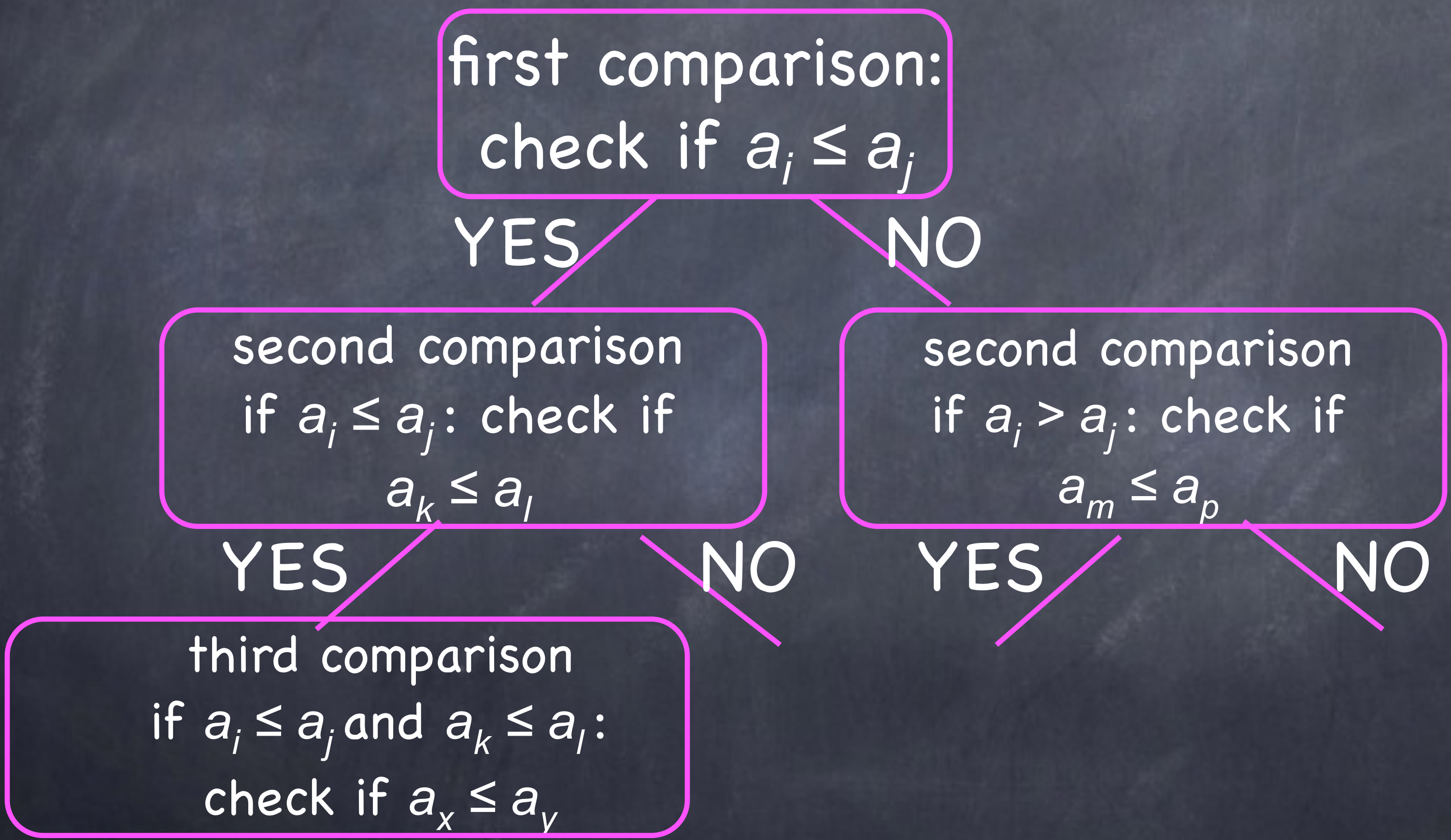
Comparison-Based Sorting

- All these algorithms are comparison-based
  - the behavior depends on relative values of keys, not exact values
  - behavior on [1,3,2,4] is same as on [9,25,23,99]
- Fastest of these algorithms was O(n log n).
- We will show that's the best you can get with comparison-based sorting.

Decision Tree

- Consider any comparison based sorting algorithm
- Represent its behavior on all inputs of a fixed size with a decision tree
- Each tree node corresponds to the execution of a comparison
- Each tree node has two children, depending on whether the parent comparison was true or false
- Each leaf represents correct sorted order for that path

Decision Tree Diagram

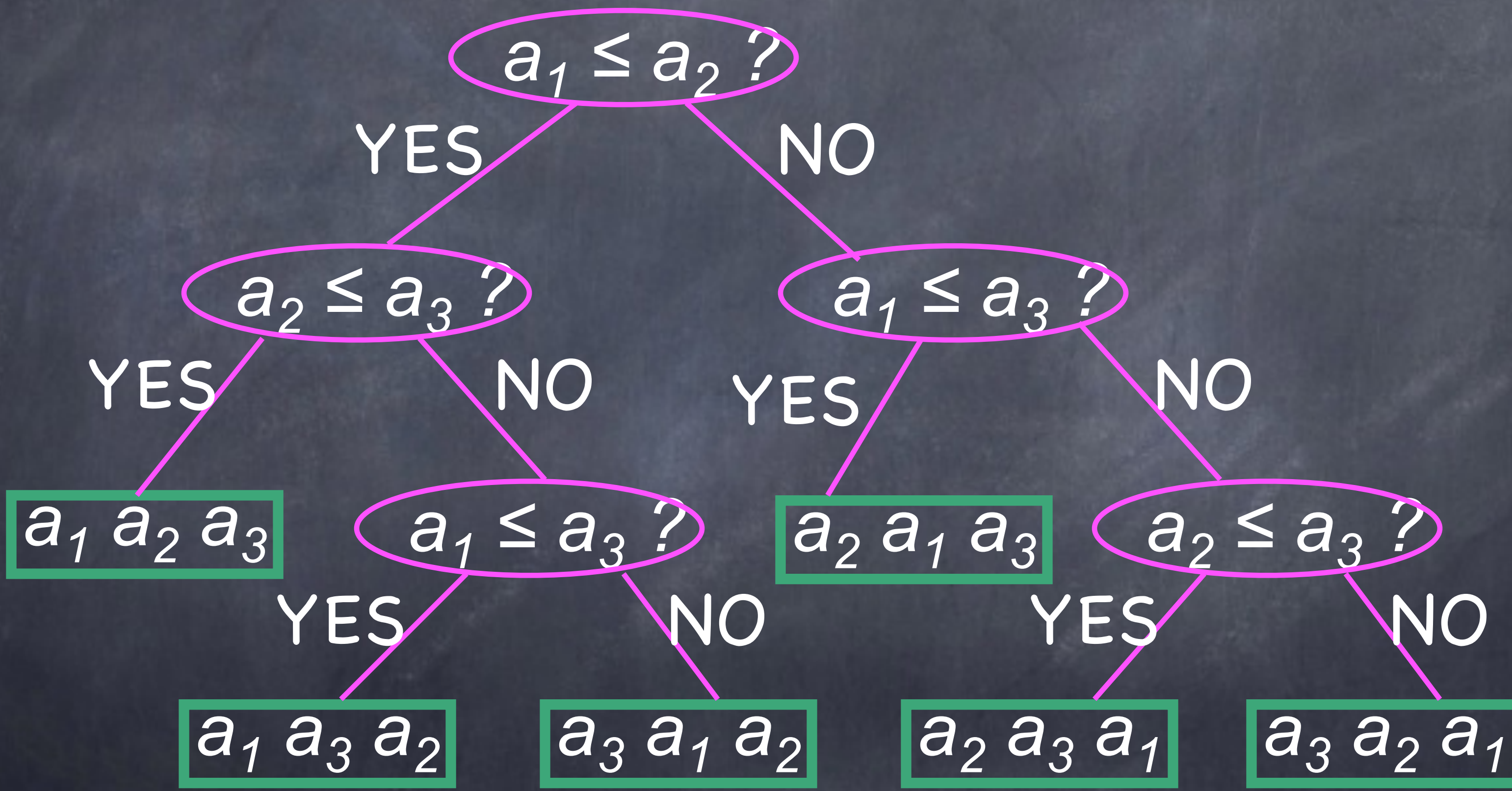


Insertion Sort

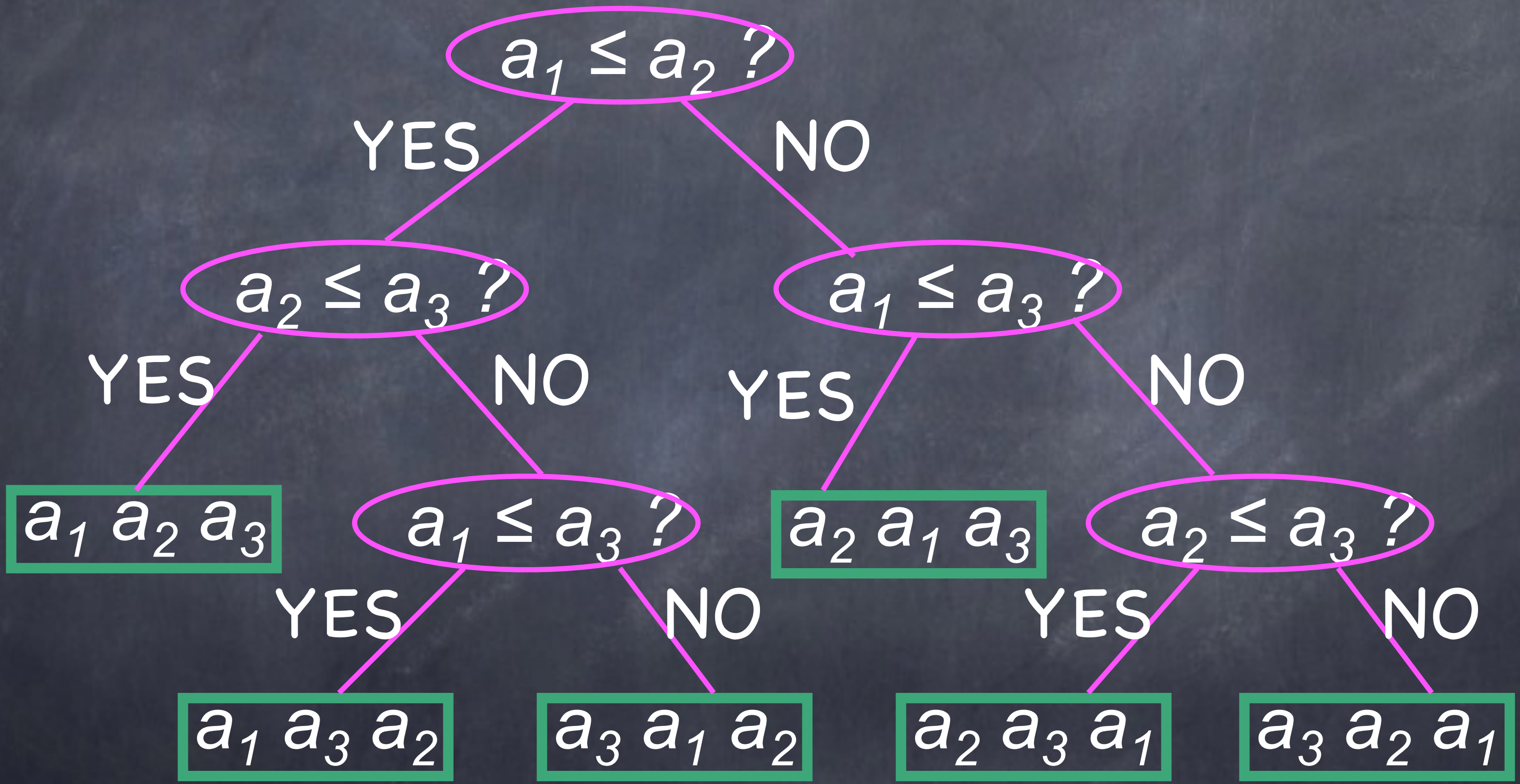
```
for j := 2 to n to
    key := a[j]
    i := j-1
    while i > 0 and a[i] > key do // insert in prev.
        a[i+1] := a[i]
        i := i -1
    endwhile
    a[i+1] := key
endfor
```

comparison

Insertion Sort for n = 3



Insertion Sort for n = 3



How Many Leaves?

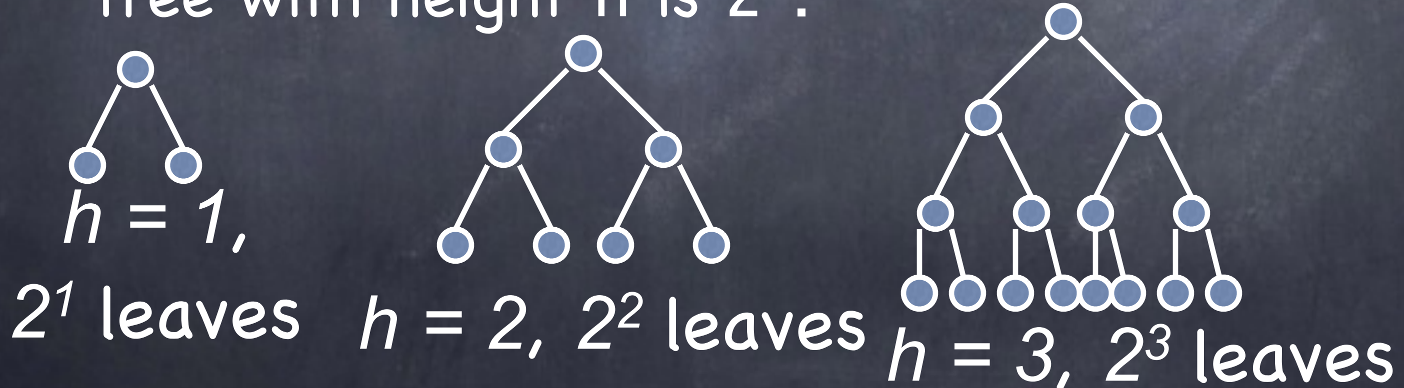
- Must be at least one leaf for each permutation of the input
  - otherwise there would be a situation that was not correctly sorted
- Number of permutations of n keys is n!.
- Idea: since there must be a lot of leaves, but each decision tree node only has two children, tree cannot be too shallow
  - depth of tree is a lower bound on running time

Key Lemma

Height of a binary tree with n! leaves is

Ω(n log n).

Proof: The maximum number of leaves in a binary tree with height h is 2^h.



Proof of Lemma

- Let h be the height of decision tree, so it has at most 2^h leaves.

- The actual number of leaves is n!, hence

$2^h \geq n!$

$h \geq \log(n!)$

$= \log(n(n-1)(n-1)...(2)(1))$

$\geq (n/2)\log(n/2)$ by algebra

$= \Omega(n \log n)$

Finishing Up

- Any binary tree with n! leaves has height $\Omega(n \log n)$.
- Decision tree for any c-b sorting alg on n keys has height $\Omega(n \log n)$.
- Any c-b sorting alg has at least one execution with $\Omega(n \log n)$ comparisons
- Any c-b sorting alg has $\Omega(n \log n)$ worst-case running time.