Deterministic and Randomized Quicksort

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Overview

- Deterministic Quicksort
- Modify Quicksort to obtain better asymptotic bound
- Linear-time median algorithm
- Randomized Quicksort

Deterministic Quicksort

```
Quicksort(A,p,r)
   if p < r then
      q := Partition(A,p,r); // rearrange A[p..r] in place
      Quicksort(A, p,q-1);
      Quicksort(A,q+1,r);
```

Divide-and-Conquer

The design of Quicksort is based on the divide-and-conquer paradigm.

a) **Divide**: Partition the array A[p..r] into two (possibly empty) subarrays A[p..q-1] and A[q+1,r] such that

- A[x] <= A[q] for all x in [p..q-1]
- A[x] > A[q] for all x in [q+1,r]

b) **Conquer**: Recursively sort A[p..q-1] and A[q+1,r]

c) **Combine**: nothing to do here

Partition

2	1	3	4	7	5	6	8
p		i					r

Select pivot (orange element) and rearrange:

- larger elements to the left of the pivot (red)
- elements not exceeding the pivot to the right (yellow)

Partition

```
Partition(A,p,r)
  x := A[r]; // select rightmost element as pivot
  i := p-1;
  for j = p to r-1 {
    if A[j] <= x then i := i+1; swap(A[i], A[j]);
  }
  swap(A[i+1],A[r]);
  return i+1;
```

Throughout the for loop:
- If p <= k <= i then A[k]<= x
- If i+1<=k <= j-1 then A[k] > x
- If k=r, then A[k] = x
- A[j..r-1] is unstructured

Partition - Loop - Example

	2	8	7	1	3	5	6	4
i	p,j							r

	2	8	7	1	3	5	6	4
	p,i	j						r

	2	8	7	1	3	5	6	4
	p,i		j					r

	2	8	7	1	3	5	6	4
	p,i			j				r

	2	1	7	8	3	5	6	4
	p	i			j			r

	2	1	3	8	7	5	6	4
	p	i				j		r

	2	1	3	8	7	5	6	4
	p		i				j	r

	2	1	3	8	7	5	6	4
	p		i					r

After the loop, the partition routine swaps the leftmost element of the right partition with the pivot element:

	2	1	3	8	7	5	6	4
	p		i					r

swap(A[i+1],A[r])

	2	1	3	4	7	5	6	8
	p		i					r

now recursively sort yellow and red parts.

Worst-Case Partitioning

The worst-case behavior for quicksort occurs on an input of length n when partitioning produces just one subproblem with n-1 elements and one subproblem with 0 elements.

Therefore the recurrence for the running time T(n) is:

$$T(n) = T(n-1) + T(0) + \theta(n) = T(n-1) + \theta(n) = \theta(n^2)$$

Perhaps we should call this algorithm pokysort?

“Better” Quicksort and Linear Median Algorithm

Best-case Partitioning

Best-case partitioning:

If partition produces two subproblems that are roughly of the same size, then the recurrence of the running time is

$$T(n) <= 2T(n/2) + \theta(n)$$

so that T(n) = O(n log n)

Can we achieve this bound?

Yes, modify the algorithm. Use a linear-time median algorithm to find median, then partition using median as pivot.

Linear Median Algorithm

Let A[1..n] be an array over a totally ordered domain.

- Partition A into groups of 5 and find the median of each group.
[You can do that with 6 comparisons]

- Make an array U[1..n/5] of the medians and find the median m of U by recursively calling the algorithm.

- Partition the array A using the median-of-medians m to find the rank of m in A. If m is of larger rank than the median of A, eliminate all elements > m. If m is of smaller rank than the median of A, then eliminate all elements <= m. Repeat the search on the smaller array.

Linear-Time Median Finding

How many elements do we eliminate in each round?

The array U contains $n/5$ elements. Thus, $n/10$ elements of U are larger (smaller) than m, since m is the median of U . Since each element in U is a median itself, there are $3n/10$ elements in A that are larger (smaller) than m.

Therefore, we eliminate $(3/10)n$ elements in each round.

Thus, the time $T(n)$ to find the median is

$T(n) <= T(n/5) + T(7n/10) + 6n/5$.

// median of U, recursive call, and finding medians of groups

Solving the Recurrence

Suppose that $T(n) <= cn$ (for some c to be determined later)

$T(n) <= c(n/5) + c(7n/10)+6n/5= c(9n/10)+6n/5$

If this is to be <= cn, then we need to have

$c(9n/10)+12n/10 <= cn$ or $12 <= c$

Suppose that $T(1) = d$. Then choose $c = \max\{12,d\}$.

An easy proof by induction yields $T(n) <= cn$.

Goal Achieved?

We can accomplish that quicksort achieves O(n log n) running time, if we use the linear-time median finding algorithm to select the pivot element.

Unfortunately, the constant in the big Oh expression becomes large, and quicksort looses some of its appeal.

Is there a simpler solution?

Randomized Quicksort

Randomized Quicksort

```
Randomized-Quicksort(A,p,r)
  if p < r then
     q := Randomized-Partition(A,p,r);
     Randomized-Quicksort(A, p,q-1);
     Randomized-Quicksort(A,p+1,r);
```

Partition

```
Randomized-Partition(A,p,r)
  i := Random(p,r);
  swap(A[i],A[r]);
  Partition(A,p,r);
```

Almost the same as Partition, but now the pivot element is not the rightmost element, but rather an element from A[p..r] that is chosen uniformly at random.

Goal

- The running time of quicksort depends mostly on the number of comparisons performed in all calls to the Randomized-Partition routine.
- Let X denote the random variable counting the number of comparisons in all calls to Randomized-Partition.

Notations

- Let z_i denote the i-th smallest element of A[1..n].
- Thus A[1..n] sorted is $\langle z_1, z_2, \ldots, z_n \rangle$.
- Let $Z_{ij} = \{z_i, \ldots, z_j\}$ denote the set of elements between z_i and z_j, including these elements.
- $X_{ij} = I\{ z_i \text{ is compared to } z_j\}$.
- Thus, X_{ij} is an indicator random variable for the event that the i-th smallest and the j-th smallest elements of A are compared in an execution of quicksort.

Number of Comparisons

Since each pair of elements is compared at most once by quicksort, the number X of comparisons is given by

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

Therefore, the expected number of comparisons is

$$\mathrm{E}[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{E}[X_{ij}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[z_i \text{ is compared to } z_j]$$

When do we compare?

When do we compare z_i to z_j?

Suppose we pick a pivot element in $Z_{ij} = \{z_i, \ldots, z_j\}$.

If $z_i < x < z_j$ then z_i and z_j will land in different partitions and will never be compared afterwards.

Therefore, z_i and z_j will be compared if and only if the first element of Z_{ij} to be picked as pivot element is contained in the set $\{z_i, z_j\}$.

Probability of Comparison

$$\begin{aligned}\Pr[z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}] \\ = \Pr[z_i \text{ is the first pivot chosen from } Z_{ij}] \\ + \Pr[z_j \text{ is the first pivot chosen from } Z_{ij}] \\ = \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}\end{aligned}$$

Expected Number of Comparisons

$$
\begin{aligned}
\mathrm{E}[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} \\
&< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \\
&= \sum_{i=1}^{n-1} O(\log n) \\
&= O(n \log n)
\end{aligned}
$$

Conclusion

It follows that the expected running time of Randomized-Quicksort is O(n log n).

It is unlikely that this algorithm will choose a terribly unbalanced partition each time, so the performance is very good almost all the time.