3SAT

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[partially based on slides by Jennifer Welch]

3SAT

Given a boolean function in conjunctive normal form such that every clause contains exactly three literals, decide whether the formula is satisfiable.

[This a special case of SAT]

Proving NP-Completeness

How do you prove that a decision problem L is NP-complete?

(1) Show that L is in NP.

(2.a) Choose an appropriate known NP-complete language L'.

(2.b) Show $L' \leq_p L$

Proof Strategy

(1) 3SAT is in NP, since we can check in polynomial time whether a given truth assignment evaluates to true.

(2.a) Choose SAT as a known NP-complete problem.

(2.b) Describe a reduction from SAT inputs to 3SAT inputs

- computable in polynomial time
- SAT input is satisfiable iff constructed 3SAT input is satisfiable

General Idea of the Reduction

We're given an arbitrary CNF formula $C = c_1 \wedge c_2 \wedge \ldots \wedge c_m$ over set of variables, where each c_i is a clause (a disjunction of literals).

We will replace each clause c_i with a conjunction of clauses c_i', and may use some extra variables. Each clause in c_i' will have exactly 3 literals. The transformed input will be conjunction of all the clauses in all the c_i'.

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \dots \vee z_k$

Case 1: $k = 1$. Use extra variables y_i^1 and y_i^2. Replace c_i with 4 clauses:

$$(z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee \neg y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^1 \vee \neg y_i^2) \wedge (z_1 \vee \neg y_i^1 \vee \neg y_i^2).$$

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 2: $k = 2$. Use extra variable y_i^1. Replace c_i with 2 clauses:

$$(z_1 \vee z_2 \vee \neg y_i^1) \wedge (z_1 \vee z_2 \vee y_i^1).$$

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 3: $k = 3$. No extra variables are needed.

Keep c_i: $(z_1 \vee z_2 \vee z_3)$

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 4: $k > 3$. Use extra variables $y_i^1, \ldots, y_i^{k-3}$. Replace c_i with k-2 clauses:

$(z_1 \vee z_2 \vee y_i^1)$ Text

$\wedge(\neg y_i^1 \vee z_3 \vee y_i^2)\wedge(\neg y_i^2 \vee z_4 \vee y_i^3)\wedge \ldots$

$\wedge(\neg y_i^{k-5} \vee z_{k-3} \vee y_i^{k-4})\wedge(\neg y_i^{k-4} \vee z_{k-2} \vee y_i^{k-3})$

$\wedge(\neg y_i^{k-3} \vee z_{k-1} \vee z_k)$

Polynomial Time Reduction

Each new formula is at most a constant times larger than the original formula, and the translation is straightforward. Therefore, the reduction is polynomial time.

Correctness of the Reduction

Show that CNF formula C is satisfiable iff the 3-CNF formula C' constructed is satisfiable.

=>: Suppose that C is satisfiable. We need to construct a satisfying truth assignment for C'.

For variables in C' that are already in C, we use same truth assignments as for C.

How should we assign T/F to the new variables?

Truth Assignment for New Variables

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 1: $k = 1$. Use extra variables y_i^1 and y_i^2. Replace c_i with 4 clauses:

$$(z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee \neg y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^1 \vee \neg y_i^2) \wedge (z_1 \vee \neg y_i^1 \vee \neg y_i^2).$$

Assign y_i's with arbitrary values, as z_1 is true

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 2: $k = 2$. Use extra variable y_i^1. Replace c_i with 2 clauses:

$(z_1 \vee z_2 \vee \neg y_i^1) \wedge (z_1 \vee z_2 \vee y_i^1)$.

Assign y_i's with arbitrary values, as $z_1 \vee z_2$ is true

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 3: $k = 3$. No extra variables are needed.

Keep c_i: $(z_1 \vee z_2 \vee z_3)$

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 4: $k > 3$. Use extra variables $y_i^1, \ldots, y_i^{k-3}$. Replace c_i with k-2 clauses:

$$(z_1 \vee z_2 \vee y_i^1)$$

$$\wedge(\neg y_i^1 \vee z_3 \vee y_i^2)\wedge(\neg y_i^2 \vee z_4 \vee y_i^3)\wedge \ldots$$

$$\wedge(\neg y_i^{k-5} \vee z_{k-3} \vee y_i^{k-4})\wedge(\neg y_i^{k-4} \vee z_{k-2} \vee y_i^{k-3})$$

$$\wedge(\neg y_i^{k-3} \vee z_{k-1} \vee z_k)$$

If z_1 or z_2 is true, set all y_i's to false, so all later clauses have a true literal.

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 4: $k > 3$. Use extra variables $y_i^1, \ldots, y_i^{k-3}$. Replace c_i with k-2 clauses:

$$(z_1 \vee z_2 \vee y_i^1)$$

$$\wedge(\neg y_i^1 \vee z_3 \vee y_i^2)\wedge(\neg y_i^2 \vee z_4 \vee y_i^3)\wedge \ldots$$

$$\wedge(\neg y_i^{k-5} \vee z_{k-3} \vee y_i^{k-4})\wedge(\neg y_i^{k-4} \vee z_{k-2} \vee y_i^{k-3})$$

$$\wedge(\neg y_i^{k-3} \vee z_{k-1} \vee z_k)$$

If z_{k-1} or z_k is the first true literal of c_i, set all y_i's to true, so all earlier clauses have a true literal.

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 4: $k > 3$. Use extra variables $y_i^1, \ldots, y_i^{k-3}$. Replace c_i with k-2 clauses:

$$(z_1 \vee z_2 \vee y_i^1)$$

$$\wedge(\neg y_i^1 \vee z_3 \vee y_i^2) \wedge (\neg y_i^2 \vee z_4 \vee y_i^3) \wedge \ldots$$

$$\wedge(\neg y_i^{k-5} \vee z_{k-3} \vee y_i^{k-4}) \wedge (\neg y_i^{k-4} \vee z_{k-2} \vee y_i^{k-3})$$

$$\wedge(\neg y_i^{k-3} \vee z_{k-1} \vee z_k)$$

If first true literal is in between, set all earlier y_i's to true and all later y_i's to false.

Correctness of Reduction

<=: Suppose the newly constructed 3SAT formula C' is satisfiable. We must show that the original SAT formula C is also satisfiable.

Use the same satisfying truth assignment for C as for C' (ignoring new variables).

Show each original clause has at least one true literal in it.

Original Clause is True

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 1: $k = 1$. Use extra variables y_i^1 and y_i^2. Replace c_i with 4 clauses:

$$c_i' = (z_1 \vee y_i^1 \vee y_i^2) \wedge (z_1 \vee \neg y_i^1 \vee y_i^2) \wedge (z_1 \vee y_i^1 \vee \neg y_i^2) \wedge (z_1 \vee \neg y_i^1 \vee \neg y_i^2).$$

If c_i' is true, then $c_i = z_1$ must be true, since one pair of literals in y_i^1 and y_i^2 must be true

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \dots \vee z_k$

Case 2: $k = 2$. Use extra variable y_i^1. Replace c_i with 2 clauses:

$c_i' = (z_1 \vee z_2 \vee \neg y_i^1) \wedge (z_1 \vee z_2 \vee y_i^1)$.

If c_i' is true, then $c_i = z_1 \vee z_2$ must be true

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 3: $k = 3$. No extra variables are needed.

Keep c_i: $(z_1 \vee z_2 \vee z_3)$

Reduction from SAT to 3SAT

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

Case 4: $k > 3$. Use extra variables $y_i^1, \ldots, y_i^{k-3}$. Replace c_i with $k-2$ clauses:

$$(z_1 \vee z_2 \vee y_i^1)$$

$$\wedge(\neg y_i^1 \vee z_3 \vee y_i^2) \wedge (\neg y_i^2 \vee z_4 \vee y_i^3) \wedge \ldots$$

$$\wedge(\neg y_i^{k-5} \vee z_{k-3} \vee y_i^{k-4}) \wedge (\neg y_i^{k-4} \vee z_{k-2} \vee y_i^{k-3})$$

$$\wedge(\neg y_i^{k-3} \vee z_{k-1} \vee z_k)$$

Suppose that there is a valuation such that c_i' is true and c_i is false. Then y_i^k must be true for all k, so the last clause in c_i' must be false, contradiction.

Conclusions

We have shown that

- 3SAT is in NP
- there exists a polynomial time reduction from SAT to 3SAT.

Therefore, 3SAT is NP-complete.