Amortized Analysis of a Binary Counter

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Binary Counter

A binary k-bit counter can be implemented with a k-element binary array. The counter is initially 0.

The only operation is increment(A), which adds 1 to the current number in the counter.

The increment operation can be implemented using the grade-school ripple-carry algorithm.

Aggregate Method

The worst case running time occurs when all k bits are flipped, so increment(A) has running time O(k).

In a sequence of n increment operations, few increments will cause that many bits to flip. Indeed,

bit 0 flips with every increment

bit 1 flips with every 2^{nd} increment

bit 2 flips with every 4^{th} increment, …

Aggregate Method

Total number of bit flips in n increment operations is

$n + n/2 + n/4 + \ldots + n/2^k < n(1/(1-1/2)) = 2n$

So total cost of the sequence is $O(n)$.

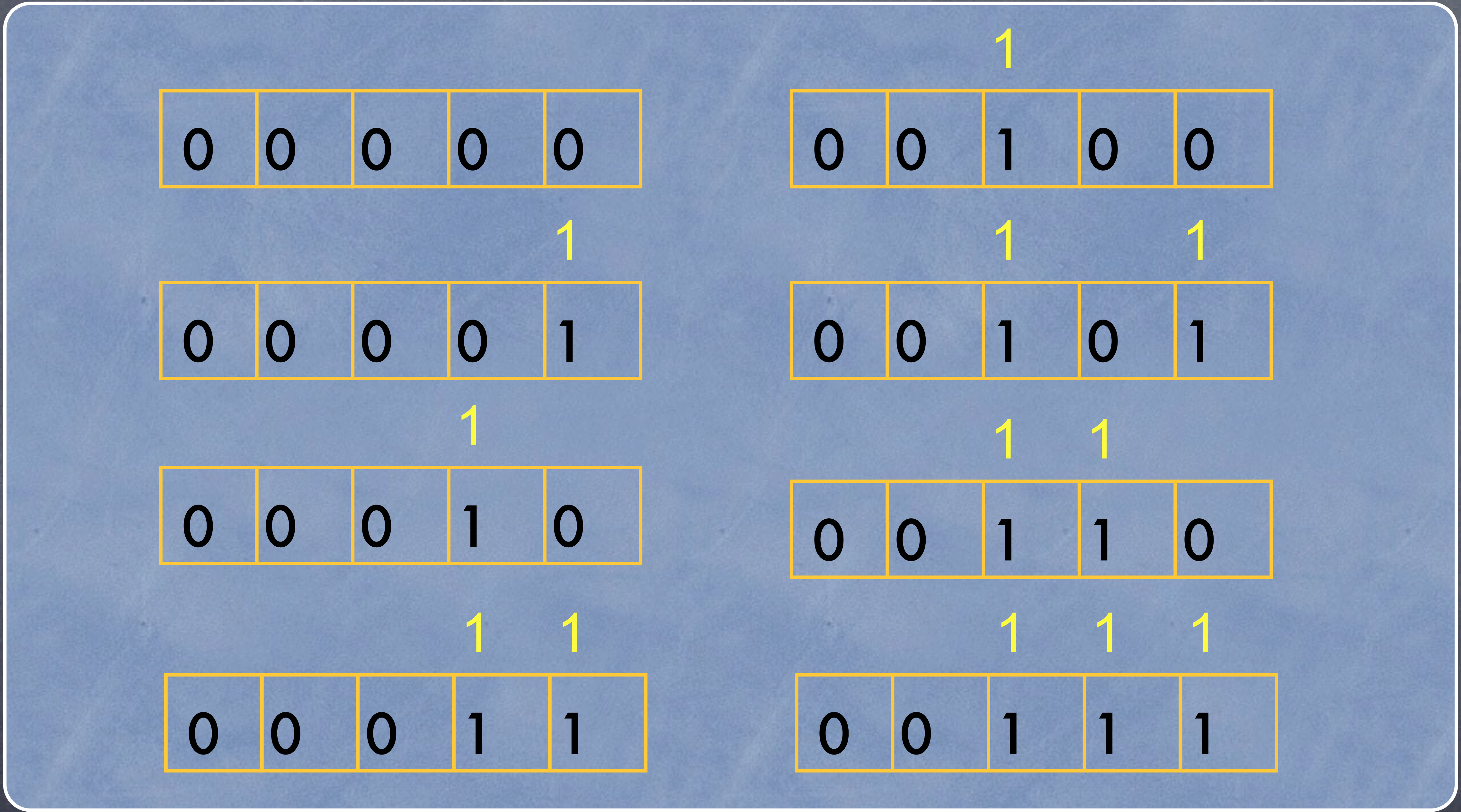
Amortized cost per operation is $O(n)/n = O(1)$.

Accounting Method for k-Bit

The actual cost for an increment operation is the number of bits flipped.

We can assign an amortized cost of 2 for each increment operation. The main idea is to use 1 to flip the bit from 0 to 1 and store 1 credit to flip it back to 0 later.

Accounting Method for K-Bit Counter



Accounting Method

All changes from 1 to 0 are paid for with previously stored credit (never go into red)

The amortized time per operation is $O(1)$

Potential Method

Potential Function B_i = number of 1s in counter after i^{th} increment.

Suppose i^{th} operation resets t_i bits.

Actual cost: $c_i = t_i + 1$

Notice that $B_i <= B_{i-1} - t_i + 1$

- if $B_i = 0$, then $B_{i-1} = t_i = k$

- if $B_i > 0$, then $B_i = B_{i-1} - t_i + 1$

Potential Method

Difference in Potentials:

$B_i - B_{i-1} <= (B_{i-1} - t_i + 1) - B_{i-1} = -t_i + 1$

Amortized cost: $c_i + B_i - B_{i-1} <= t_i + 1 - t_i + 1 = 2$

Advantage of the potential method: We can use it to analyze counters that do not start from 0, see CLRS.