# Asymptotic Analysis 4: Asymptotic Lower Bounds

Andreas Klappenecker and Hyunyoung Lee

Texas A&M University

Let $f$ and $g$ be functions from the set of natural numbers to the set of real numbers. We say that $g$ is an **asymptotic lower bound** to $f$ and write $f \in \Omega(g)$ if and only if there exists a positive constant $c$ and a natural number $n_0$ such that

$$c|g(n)| \leqslant |f(n)|$$

holds for all $n \geqslant n_0$. This formalizes the notion that $f(n)$ grows at least as fast as a constant multiple of $g(n)$ for large $n$.

This asymptotic lower bound is related to the asymptotic upper bound in the following way.

**Proposition**

*Let $f$ and $g$ be functions from the set of natural numbers to the set of real numbers. We have $f \in \Omega(g)$ if and only if $g \in O(f)$.*

# Proof

We have $f \in \Omega(g)$ if and only if there exists a positive constant $c$ and a natural number $n_0$ such that $c|g(n)| \leqslant |f(n)|$ holds for all $n \geqslant n_0$. Dividing both sides by $c$ shows that there exist a positive constant $C = 1/c$ and a natural number $n_0$ such that $|g(n)| \leqslant \frac{1}{c}|f(n)| = C|f(n)|$ holds for all $n \geqslant n_0$. However, this is nothing but the definition of $g \in O(f)$.

Let $f$ and $g$ be functions from the set of natural numbers to the set of real numbers. We say that $g$ is an **strict asymptotic lower bound** to $f$ and write $f \in \omega(g)$ if and only if for all positive constants $c$ there exists a natural number $n_0$ such that

$$c|g(n)| \leqslant |f(n)|$$

holds for all $n \geqslant n_0$.

**Example**

The function $n^2$ is in $\omega(n)$, since for a given positive constant $c$, the inequality $cn = c|n| \leqslant |n^2| = n^2$ holds for all natural numbers $n \geqslant c$.

On the other hand, $n$ is not in $\omega(n)$, since there does not exist any natural number $n$ for which $2n = 2|n| \leqslant |n| = n$ holds.

**Proposition**

*Let $f$ and $g$ be functions from the set of natural numbers to the set of real numbers, and assume that $g$ is eventually nonzero. Then we have $f \in \omega(g)$ if and only if*

$$\lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = \infty.$$

**Proof.**

We have $f \in \omega(g)$ if and only if for all positive constants $c$ there exists a natural number $n_0$ such that $c \leqslant |f(n)|/|g(n)|$ holds for all $n \geqslant n_0$, so $|f(n)|/|g(n)|$ grows without bound. By definition of the limit, this is equivalent to

$$\lim_{n \to \infty} \frac{|f(n)|}{|g(n)|} = \infty,$$

which proves the claim. □