Asymptotic Analysis 3: Asymptotic Upper Bounds

Andreas Klappenecker and Hyunyoung Lee

Texas A&M University

Asymptotic Upper Bound

Let f and g be functions from the natural numbers to the real numbers. We say that g is an **asymptotic upper bound** for f and write $f \in O(g)$ if and only if there exists a positive real constant C and a natural number n_0 such that

$$|f(n)| \leqslant C|g(n)|$$

holds for all $n \geqslant n_0$.

Proposition

Let f be a function from the natural numbers to the real numbers, and g an eventually nonzero function from the natural numbers to the real numbers. Then $f(n) \in O(g(n))$ if and only if

$$\limsup_{n \to \infty} \frac{|f(n)|}{|g(n)|} < \infty.$$

Corollary

Let f and g be functions from the set of natural numbers to the of real numbers. If the limit

$$\lim_{n \to \infty} \frac{|f(n)|}{|g(n)|}$$

exists and is finite, then $f \in O(g)$.

We say that g is a **strict asymptotic upper bound** for f and write $f \in o(g)$ if and only if for every $\epsilon > 0$ there exists a natural number n_ϵ such that

$$|f(n)| \leqslant \epsilon |g(n)|$$

holds for all $n \geqslant n_\epsilon$. By definition, $f \in o(g)$ implies that $f \in O(g)$.

Proposition

Let f and g be functions from the set of natural numbers to the set of real numbers such that g is eventually nonzero. Then $f \in o(g)$ if and only if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \tag{1}$$

holds.

Proof

Suppose that (1) holds. By definition of the limit, this means that for any $\epsilon > 0$ there exists a natural number n_ϵ such that

$$\left|\frac{f(n)}{g(n)}\right| < \epsilon$$

holds for all $n \geqslant n_\epsilon$. This is equivalent to the condition that for each $\epsilon > 0$ there exists an n_ϵ such that

$$|f(n)| \leqslant \epsilon |g(n)|$$

holds for all $n \geqslant n_\epsilon$. In other words, (1) is equivalent to $f \in o(g)$.

Corollary

Let f and g be functions from the set of real natural numbers to the set of real numbers. Suppose that $f = o(g)$. Then

$$g + f = O(g).$$

Example

Since $n^{1000} + n^2 + 1 \in o(\exp(n))$, we have

$$\exp(n) + n^{1000} + n^2 + 1 \in O(\exp(n)).$$

Example

Recall that the Harmonic number satisfies

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - E(n),$$

where γ is Euler's constant $\gamma \approx 0.5772156649$, and the value of the error term $E(n)$ is in the range $0 < E(n) < 1/(252n^6)$. It follows that

$$H_n = \log n + \gamma + O\left(\frac{1}{n}\right).$$

Rules

Constants

If c is a nonzero constant, then

$$cO(f(n)) = O(f(n)), \tag{2}$$
$$O(cf(n)) = O(f(n)). \tag{3}$$

Idempotency

The Big Oh operator is idempotent, meaning that

$$O(O(f(n)) = O(f(n)). \tag{4}$$

Rules

Multiplications

The multiplication of Big Oh expressions follows the rules

$$O(f(n))O(g(n)) = O(f(n)g(n)), \tag{5}$$

$$O(f(n)g(n)) = f(n)O(g(n)). \tag{6}$$

Absorbtion.

We can simplify Big Oh expressions using the rule

$$O(f(n)) + O(g(n)) = O(g(n)) \text{ provided that } f(n) = O(g(n)). \tag{7}$$

Rules

Powers

For all positive integers k, we have

$$(f(n) + g(n))^k = O\big((f(n))^k\big) + O\big((g(n))^k\big). \tag{8}$$

Linear Combinations

If $f(n) = O(h(n))$ and $g(n) = O(h(n))$, then

$$af(n) + bg(n) = O(h(n)) \quad \text{for all } a, b \in \mathbf{C}. \tag{9}$$

Rules

Swap

The next rule allows you to swap Big Oh terms.

$$\text{If } f(n) = g(n) + O(h(n)) \text{ then } g(n) = f(n) + O(h(n)). \tag{10}$$