Asymptotic Analysis 2: Asymptotically Tight Bounds

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Motivation

The asymptotic equality is often a bit too strict. Sometimes it is desirable to relax the constraints and consider

(a) **the growth up to a constant factor** and

(b) **without the need for the existence of a limit**.

Asymptotically Tight Bounds

Let f and g denote functions from the natural numbers to the real numbers. We say that f and g have the **same order of growth** and write $f \asymp g$ or $f \in \Theta(g)$ if and only if there exist positive real constants c and C and a natural number n_0 such that

$$c|g(n)| \leqslant |f(n)| \leqslant C|g(n)|$$

holds for all $n \geqslant n_0$.

Notations

The notation $f \asymp g$ goes back to Hardy and is popular in mathematics. Computer scientists like to express this in the form $f \in \Theta(g)$, where

$$\Theta(g) = \{f : \mathbf{N} \to \mathbf{R} \mid f \asymp g\}$$

is the set of functions that have the same order of growth as g. If $f \in \Theta(g)$ or $f \asymp g$, then we also say that g is an **asymptotically tight bound** for f.

Proposition

Let f and g be functions from the set of natural numbers to the set of real numbers. If g is a positive function and the limit

$$d = \lim_{n \to \infty} \frac{|f(n)|}{|g(n)|}$$

exists and is a nonzero real number d, then $f \in \Theta(g)$.

Proof

It follows from the definition of the limit that for each $\epsilon > 0$ there exists a natural number n_ϵ such that

$$d - \epsilon \leqslant \frac{|f(n)|}{|g(n)|} \leqslant d + \epsilon$$

for all $n \geqslant n_\epsilon$. In other words, for the constants $c = d - \epsilon$ and $C = d + \epsilon$ there exists an n_ϵ such that $c|g(n)| \leqslant |f(n)| \leqslant C|g(n)|$ holds for all $n \geqslant n_\epsilon$, which proves $f \in \Theta(g)$.

Corollary

If two functions f and g are asymptotically equal, $f \sim g$, then they have the same order of growth, that is, $f \asymp g$.

Example

Let $f(n) = (2 + (-1)^n)n^2$ and $g(n) = n^2$. Then the limit

$$\lim_{n\to\infty} \frac{|f(n)|}{|g(n)|}$$

does not exist, as the quotient fluctuates between 3 and 1, but

$$|g(n)| \leqslant |f(n)| \leqslant 3|g(n)|$$

holds for all $n \geqslant 1$; hence, $f \in \Theta(g)$.

We can characterize $f \in \Theta(g)$ using limit superior and limit inferior from calculus. We recall the relevant terminology.

Let f be a function from the set of natural numbers to the set of real numbers. The real number u is an **upper accumulation point** of f if and only if the following two conditions are met:

U1. For each $\epsilon > 0$ there exist infinitely many natural numbers n such that $f(n) > u - \epsilon$,

U2. For each $\epsilon > 0$ there exist at most finitely many natural numbers such that $f(n) > u + \epsilon$.

If an upper accumulation point of f exists, then it is unique.

Limit Superior

The function f is called **bounded above** if and only if there exists a real number u such that $f(n) \leqslant u$ holds for all natural numbers n.
If f has an upper accumulation point, then it is bounded above.
The **limit superior** of f is defined as

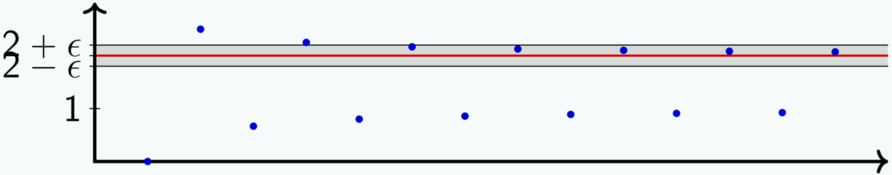
$$\limsup_{n\to\infty} f(n) = \begin{cases} +\infty & \text{if } f \text{ is not bounded above,} \\ u & \text{if the upper accumulation point } u \text{ of } f \text{ exists,} \\ -\infty & \text{otherwise.} \end{cases}$$

Unlike the limit of f, the limit superior of f always exists.

Example

Let $f(n)$ denote the function given by

$$f(n) = \begin{cases} 2 + 1/n & \text{if } n \text{ is even} \\ 1 - 1/n & \text{if } n \text{ is odd} \end{cases}$$



We can conclude that $\limsup_{n\to\infty} f(n) = 2$ and $\liminf_{n\to\infty} f(n) = 1$.

Proposition

Let f be a function from the natural numbers to the real numbers, and g an eventually nonzero function from the natural numbers to the real numbers. Then $f \in \Theta(g)$ if and only if

$$\liminf_{n\to\infty} \frac{|f(n)|}{|g(n)|} > 0 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} < \infty.$$

Proof

If $f \in \Theta(g)$, then there exists a positive constants c and C and a natural number n_0 such that $c \leqslant |f(n)|/|g(n)| \leqslant C$ holds for all $n \geqslant n_0$. This implies that

$$\liminf_{n\to\infty} \frac{|f(n)|}{|g(n)|} \geqslant c > 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} \leqslant C < \infty$$

hold.

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hold.
Conversely, suppose that f and g are functions satisfying

$$c := \liminf_{n\to\infty} |f(n)|/|g(n)| > 0 \quad \text{and} \quad C := \limsup_{n\to\infty} |f(n)|/|g(n)| < \infty.$$

By definition of the limit superior and inferior, for any ϵ in the range $0 < \epsilon < c$ there exists a natural number n_0 such that

$$0 < c - \epsilon \leqslant \frac{|f(n)|}{|g(n)|} \leqslant (C + \epsilon)$$

holds for all $n \geqslant n_0$. Multiplying these inequalities by $|g(n)|$ shows that $f \in \Theta(g)$ holds.