Asymptotic Analysis 1: Limits and Asymptotic Equality

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Motivation

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First, let us recall the notion of a limit.

Limit

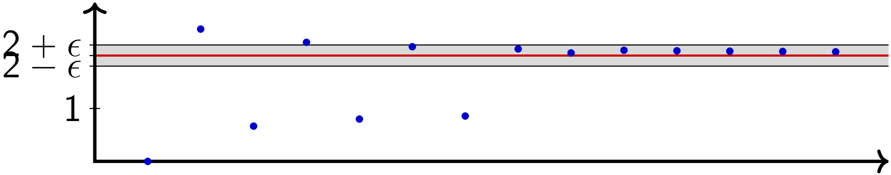
Given a function $f\colon \mathbf{N}_0 \to \mathbf{R}$, we say that f converges to the limit $L \in \mathbf{R}$ as $n \to \infty$, and write

$$\lim_{n\to\infty} f(n) = L,$$

if and only if for each $\epsilon > 0$ there exists an $n_\epsilon \in \mathbf{N}_0$ such that

$$|f(n) - L| < \epsilon$$

holds for all $n \geqslant n_\epsilon$.



2 + ϵ
2
2 − ϵ
1

Limit

Given a function $f\colon \mathbf{N}_0 \to \mathbf{R}$, we say that f tends to ∞ as $n \to \infty$, and write

$$\lim_{n\to\infty} f(n) = \infty,$$

if and only if for each real number B there exists an $n_B \in \mathbf{N}_0$ such that $f(n) > B$ for all $n \geqslant n_B$.

Proposition

Suppose that we are given functions $f, g, h : \mathbf{N}_0 \to \mathbf{R}$ *such that there exists a positive integer* n_0 *such that for all* $n \geqslant n_0$*, the inequality chain*

$$f(n) \leqslant g(n) \leqslant h(n)$$

holds, and

$$\lim_{n\to\infty} f(n) = L = \lim_{n\to\infty} h(n).$$

Then $\lim_{n\to\infty} g(n)$ *exists and has the same limit*

$$\lim_{n\to\infty} g(n) = L.$$

Let f and g be functions from the set of natural numbers to the set of real numbers. We write $f \sim g$ and say that f is **asymptotically equal** to g if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

holds.

Asymptotic Equality

By definition of the limit this means that for each $\epsilon > 0$ there exists a natural number n_ϵ such that

$$\left| \frac{f(n)}{g(n)} - 1 \right| < \epsilon \tag{1}$$

holds for all $n \geqslant n_\epsilon$.

Asymptotic Equality

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holds for all $n \geqslant n_\epsilon$.

One way to interpret the inequality (1) is that two functions f and g are asymptotically equal if and only if the relative error $(f(n) - g(n))/g(n)$ between these functions vanishes for large n. Essentially, this means that the functions f and g have the same growth for large n.

Proposition

The n-th Harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is asymptotically equal to the natural logarithm $\ln n$,

$$H_n \sim \ln n.$$

Since the inequalities $\ln(n+1) \leqslant H_n \leqslant 1 + \ln n$ hold, dividing by $\ln n$ and taking the limit yields for the logarithmic terms

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{1+\ln n}{\ln n} = 1,$$

where we used l'Hôpital's rule in the calculation of the first limit. Thus, it follows from the squeeze theorem for limits that

$$\lim_{n\to\infty} \frac{H_n}{\ln n} = 1,$$

which proves that $H_n \sim \ln n$. In other words, the Harmonic numbers grow like the natural logarithm for large n.

Example

The Stirling approximation yields

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

One advantage of the asymptotic equality $\sim$ is that the expression can be simplified quite a bit. The next proposition illustrates this in the case of polynomials.

Proposition

Let $p(x) = \sum_{k=0}^{m} a_k x^k$ be a nonzero polynomial of degree m with real coefficients. Then $p(x)$ is asymptotically equal to its leading term,

$$p(x) \sim a_m x^m.$$

Criterion

Proposition

Let c be a positive real number. Let f be a continuously differentiable function from the set of positive real numbers to the set of real numbers such that its derivative f' is monotonic, nonzero, and satisfies

$$\lim_{n\to\infty} f'(n+c)/f'(n) = 1.$$

Then

$$f(n+c) - f(n) \sim cf'(n).$$

Proof

By the mean value theorem of calculus, there exists a real number θ in the range $0 \leqslant \theta \leqslant c$ such that

$$f(n+c) - f(n) = (n+c-n)f'(n+\theta) = cf'(n+\theta).$$

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$$f(n+c) - f(n) = (n+c-n)f'(n+\theta) = cf'(n+\theta).$$

If f' is monotonically increasing (or monotonically decreasing), then

$$cf'(n) \underset{(\geqslant)}{\leqslant} f(n+c) - f(n) \underset{(\geqslant)}{\leqslant} cf'(n+c).$$

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$$cf'(n) \underset{(\geqslant)}{\leqslant} f(n+c) - f(n) \underset{(\geqslant)}{\leqslant} cf'(n+c).$$

Dividing by $cf'(n)$ yields by assumption

$$\lim_{n\to\infty} \frac{cf'(n)}{cf'(n)} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{cf'(n+c)}{cf'(n)} = 1.$$

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Therefore, by the squeeze theorem for limits, we have

$$\lim_{n\to\infty} \frac{f(n+1) - f(n)}{cf'(n)} = 1,$$

which proves our claim.

Example

Let c be a positive constant. Then

$$\sqrt{n+c} - \sqrt{n} \sim \frac{c}{2\sqrt{n}}.$$

Indeed, if we set $f(x) = \sqrt{x}$, then f is a continuously differentiable function on the positive real numbers. Its derivative $f'(x) = 1/(2\sqrt{x})$ is nonzero, monotonically decreasing, and satisfies $\lim_{n\to\infty} f(n+c)/f(n) = 1$. The claim follows from the previous proposition.