NP-Completeness of Combinatorial Problems

Completing Partial Latin Squares, Sudoku, Futoshiki

Andreas Klappenecker

Partial Latin Squares

A partial Latin square of order n is an nxn array such that

- each entry is either empty or contains a number from {1,...,n}
- numbers in each row are distinct
- numbers in each column are distinct

A partial Latin square without empty cells is called a Latin square.

Optical Routing

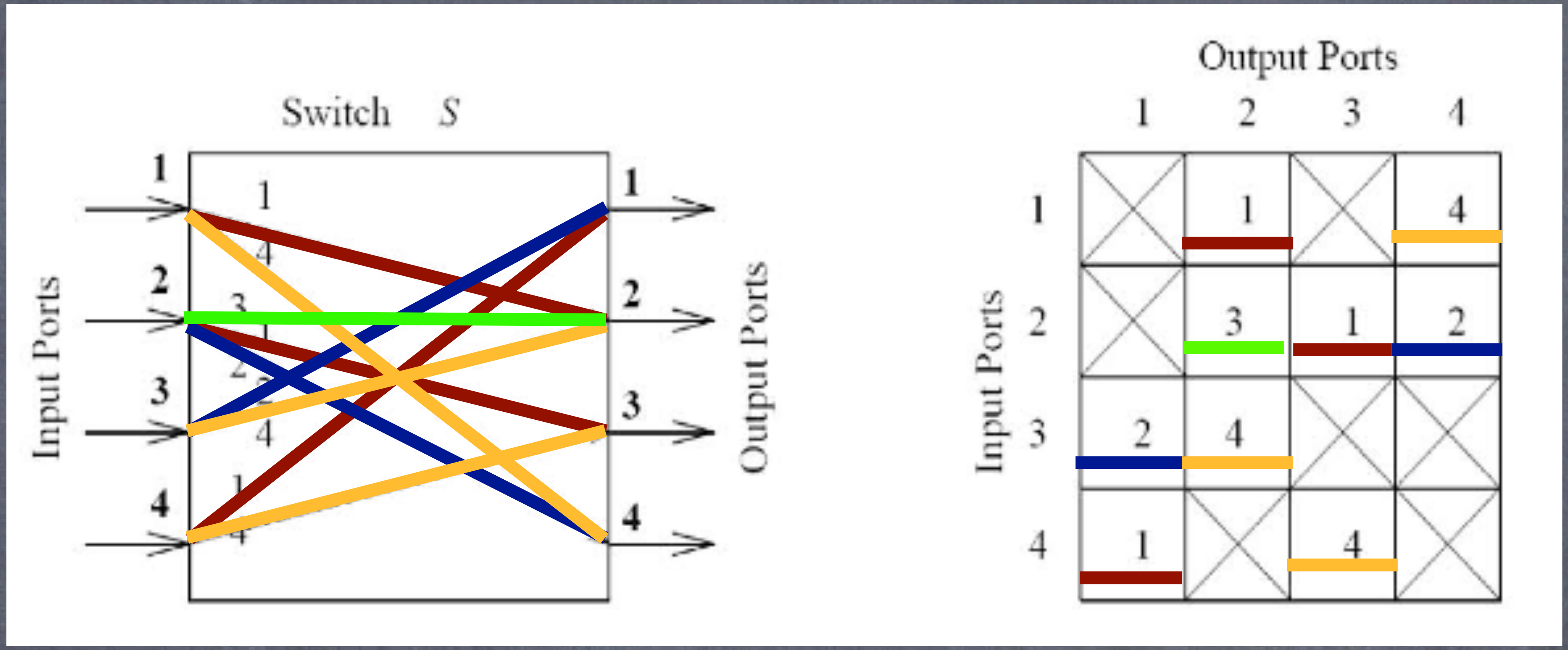
- An optical network consists of routing nodes that are connected by optical fibers
- The large bandwidth of the optical fibers is subdivided into several channels using light of different wavelengths (wavelength division multiplexing).
- Hardware design for the switches imposes constraints:

(a) # input ports = # output ports,

(b) at most one wavelength is used for the channel connecting input port and output port,

(c) wavelength switching matrix w(input a, output b) is partial latin square

Optical Routing



Switch array is a partial Latin square. Can we complete it to a Latin square to better utilize the switch?

Completing Partial Latin Squares

CPLS: Can a partial Latin square be completed to a Latin square?

Defect Graph

Given a partial Latin square P, its defect graph G(P) is a graph with vertex set $V = R \cup C \cup E$ and the edge set F as follows:

- $R = \{ r_i \mid$ row i contains an empty square $\}$
- $C = \{ c_j \mid$ column j contains an empty square $\}$
- $E = \{ e_k \mid$ element k appears in fewer than n squares $\}$.

(1) (r_i, c_j) in F if the (i, j) square of P is empty,

(2) (r_i, e_k) in F if row i does not contain element k,

(3) (c_j, e_k) in F if column j does not contain element k.

Triangle $\{r_i, c_j, e_k\}$ specifies potential completion of cell (i,j) by k

Example

1	1	2	3	4
2	3	4		2
3	2	3		1
4	4	1	2	3
	1	2	3	4

Edges:

{ r_2, c_3 }, { r_3, c_3 }

{ r_2, e_1 }, { r_3, e_4 }

{ c_3, e_1 }, { c_3, e_4 }

Tripartite Graphs

An undirected graph G=(V,E) is called tripartite if and only if there exist three independent sets V_1, V_2, V_3 that partition V.

The defect graph of a partial Latin square is tripartite (with row, column, and edge defects as independent sets: R, C, and E).

Partitioning Tripartite Graphs

A triangle partition of a graph G=(V,E) is a partition of the edge set E into sets containing three edges that form a triangle.

G(P) has a triangle-partition if and only if the partial Latin square P can be completed.

[Exercise: Prove it!]

Example

1
2
3
4

1	2	3	4
3	4		2
2	3		1
4	1	2	3

1	2	3	4

Edges:

{ r_2, c_3 }, { r_3, c_3 }

{ r_2, e_1 }, { r_3, e_4 }

{ c_3, e_1 }, { c_3, e_4 }

Triangle partition:

T_1: { r_2, c_3 }, { r_2, e_1 }, { c_3, e_1 }

T_2: { r_3, c_3 }, { r_3, e_4 }, { c_3, e_4 }

NPC of Triangle Partitions

TTP: The problem of deciding whether a given tripartite graph has a triangle partition is NP-complete.

Study the argument given in

[Charles Colburn, Discrete Applied Math 8(1):25-30, 1984]

The reduction 3SAT $\leqslant_p$ TTP is used, and is based on work by Hoyler.

NPC of Completing Partial Latin Squares

Colburn showed

$TTP \leqslant_p CLPS$

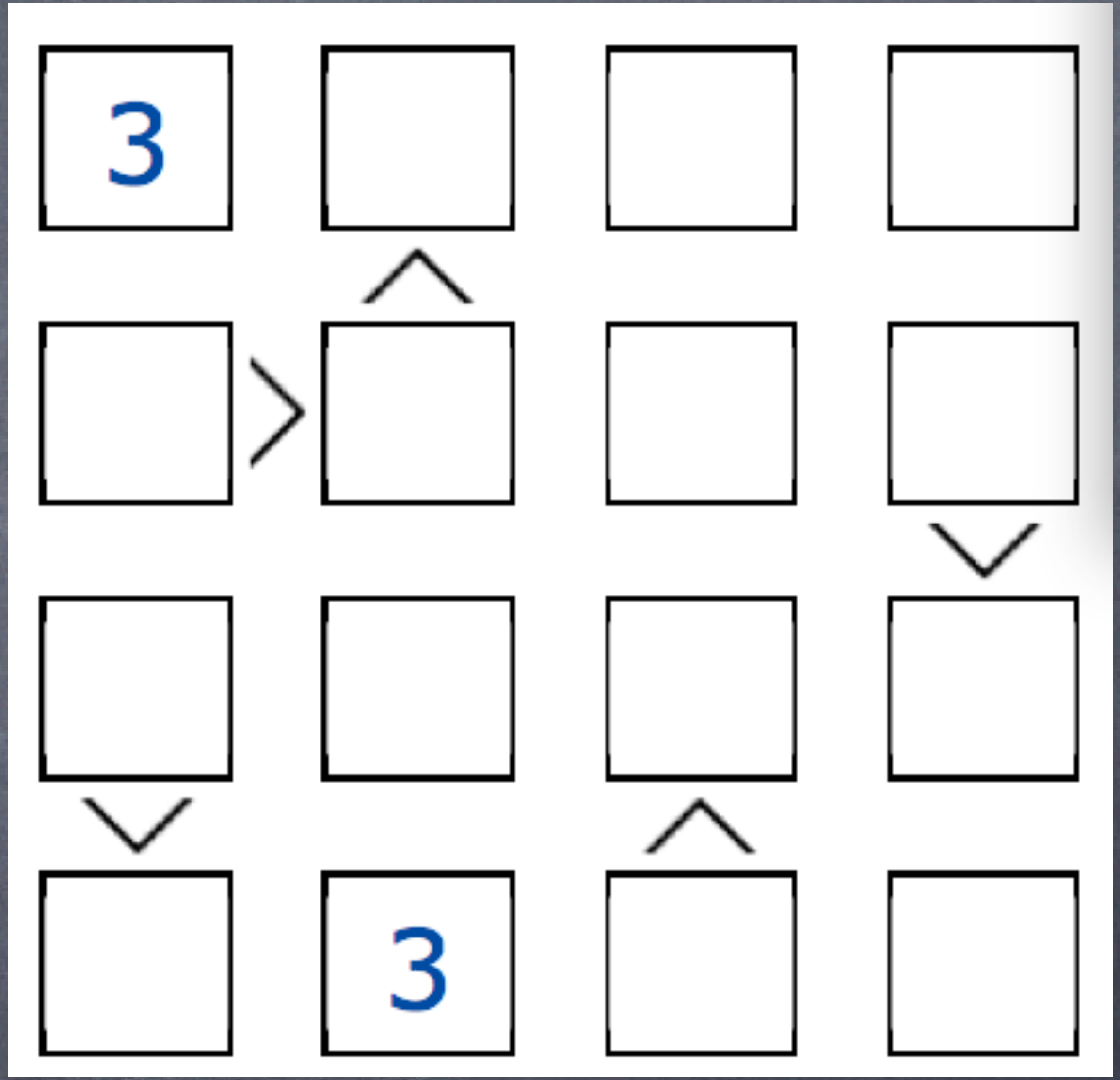
Since CLPS is in NP, it follows that completing Partial Latin Squares is NP-complete.

Exercise: Futoshiki

Recall that a Futoshiki puzzle is a partial Latin square with additional inequality constraints.

FUTOSHIKI: The problem to decide whether a given Futoshiki puzzle can be solved.

Show that FUTOSHIKI is NP-complete.



Exercise: Sudoku

9	1				4	8		5
	8	2			3			7
	3		6					
			8	4		3	6	9
8				5				1
2	6	3		9	7			
					5		8	
7			4			1	5	
6		8	9				7	3

Show that deciding whether a $n^2 \times n^2$ Sudoku problem can be solved is NP-complete.

Hints

For the Sudoku example, I suggest the following:

- Choose elements from the range $[0..n^2-1]$
- Represent them as 2 digit numbers in base n
- Study how some canonical $n^2 x n^2$ Sudoku solutions look like in this representation
- A natural choice is CPLS $\leqslant_p$ SUDOKU, but the additional constraints of SUDOKU must be taken into account!

Conclusion

Many combinatorial problems are NP-hard. We discovered that

- completing the optical routing table is NP-hard
- completing partial Latin squares is NP-hard
- Futoshiki is NP-hard
- Sudoku is NP-hard

Arguments used: 3SAT $\leqslant_p$ TTP $\leqslant_p$ CPLS $\leqslant_p$ FUTOSHIKI, SUDOKU