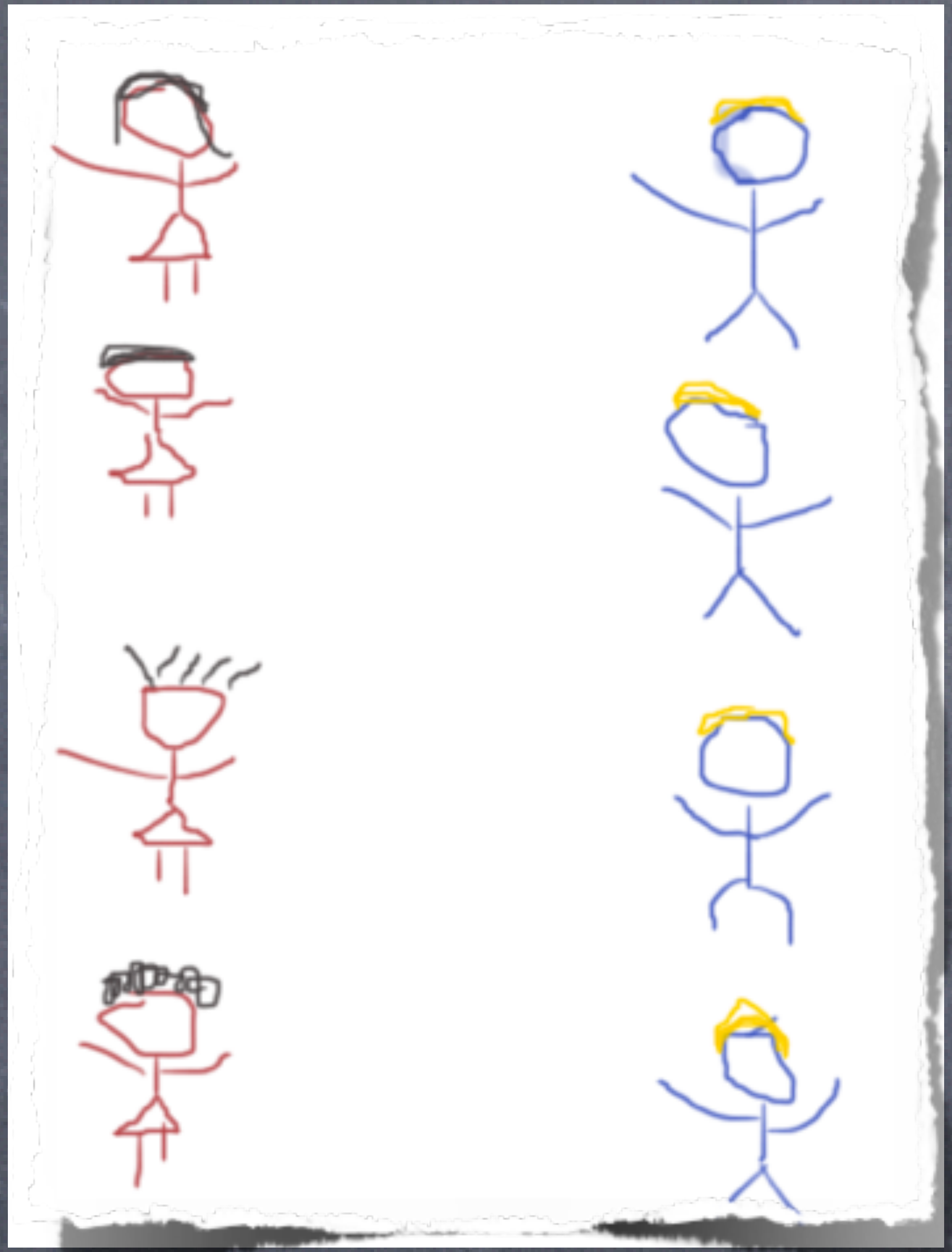
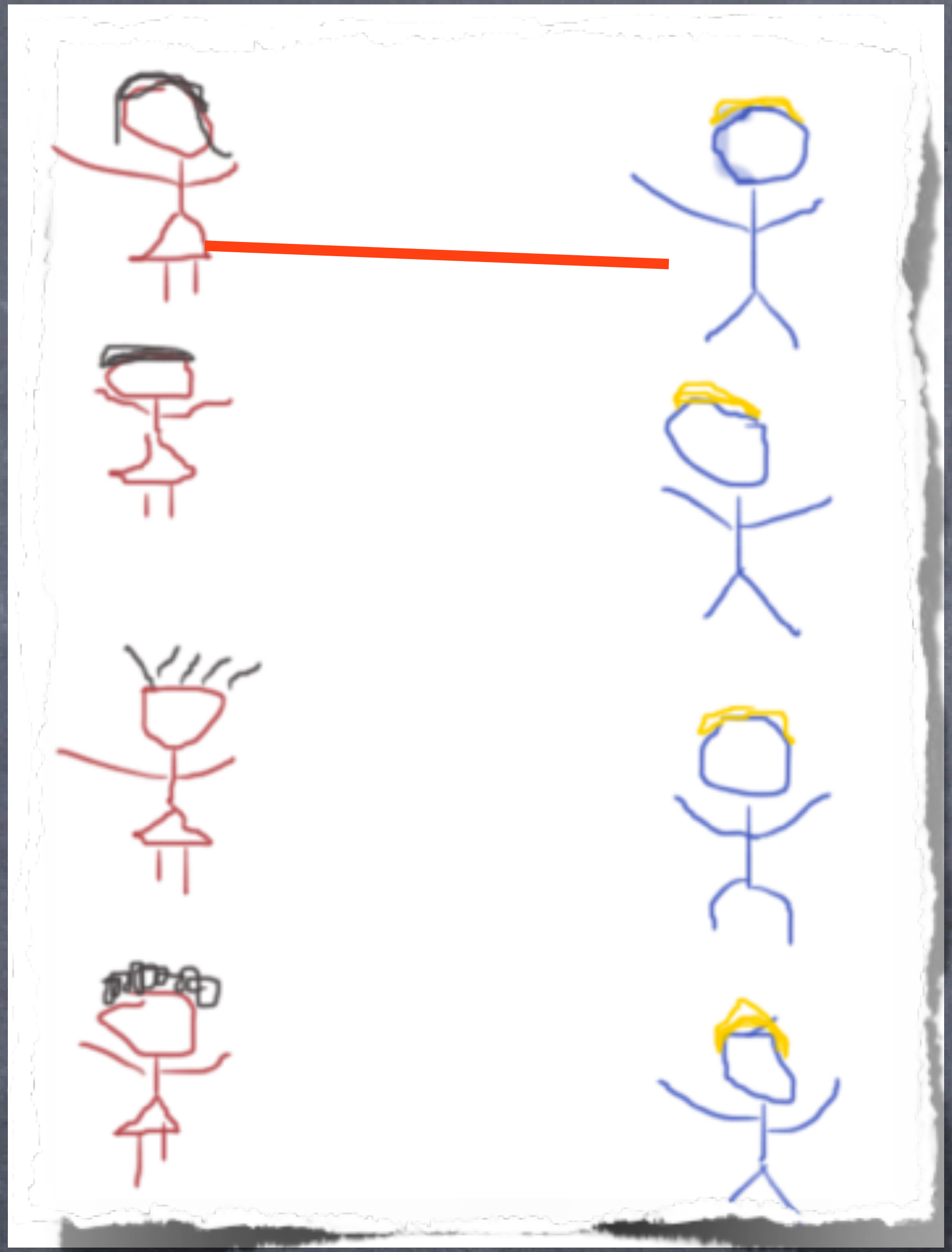
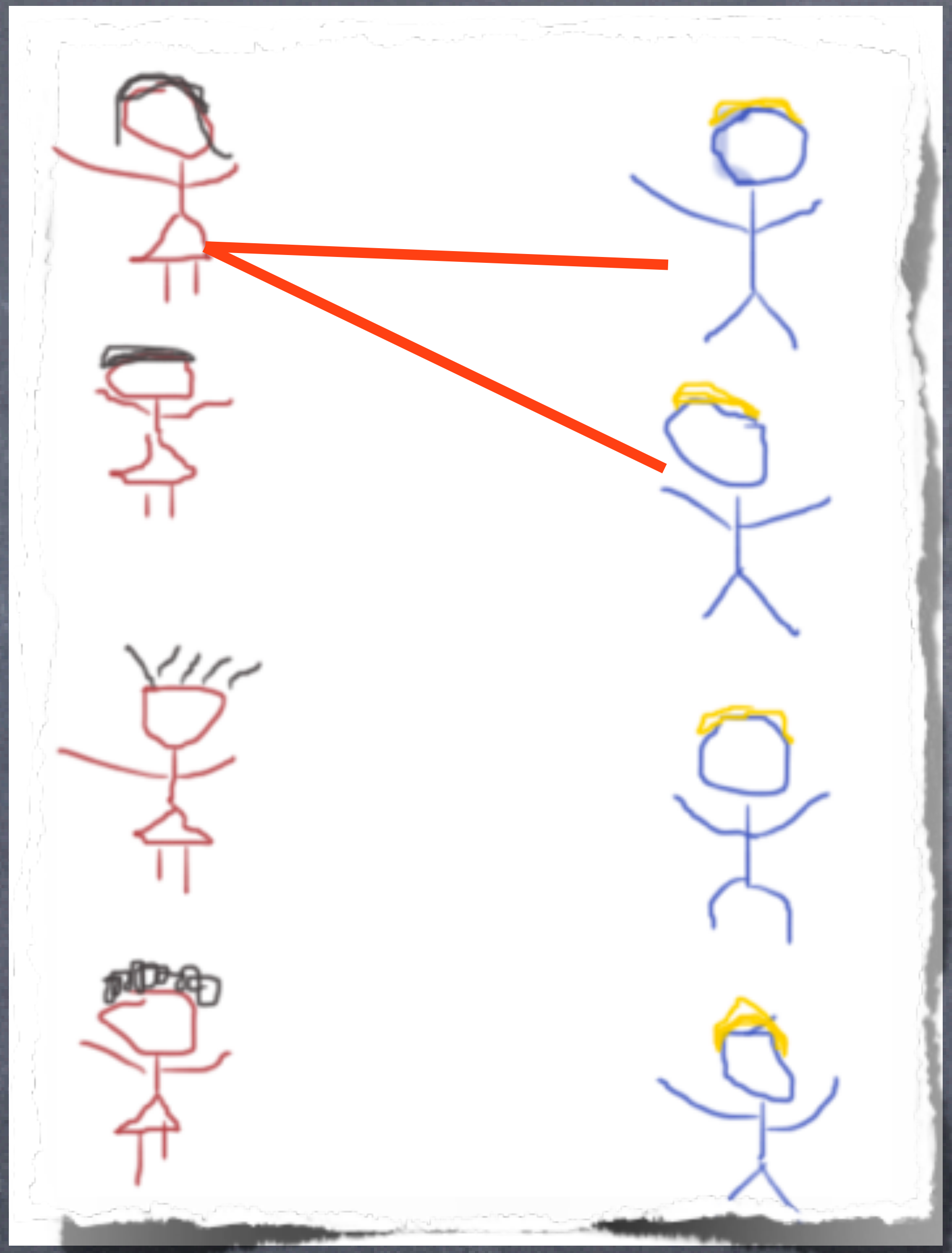
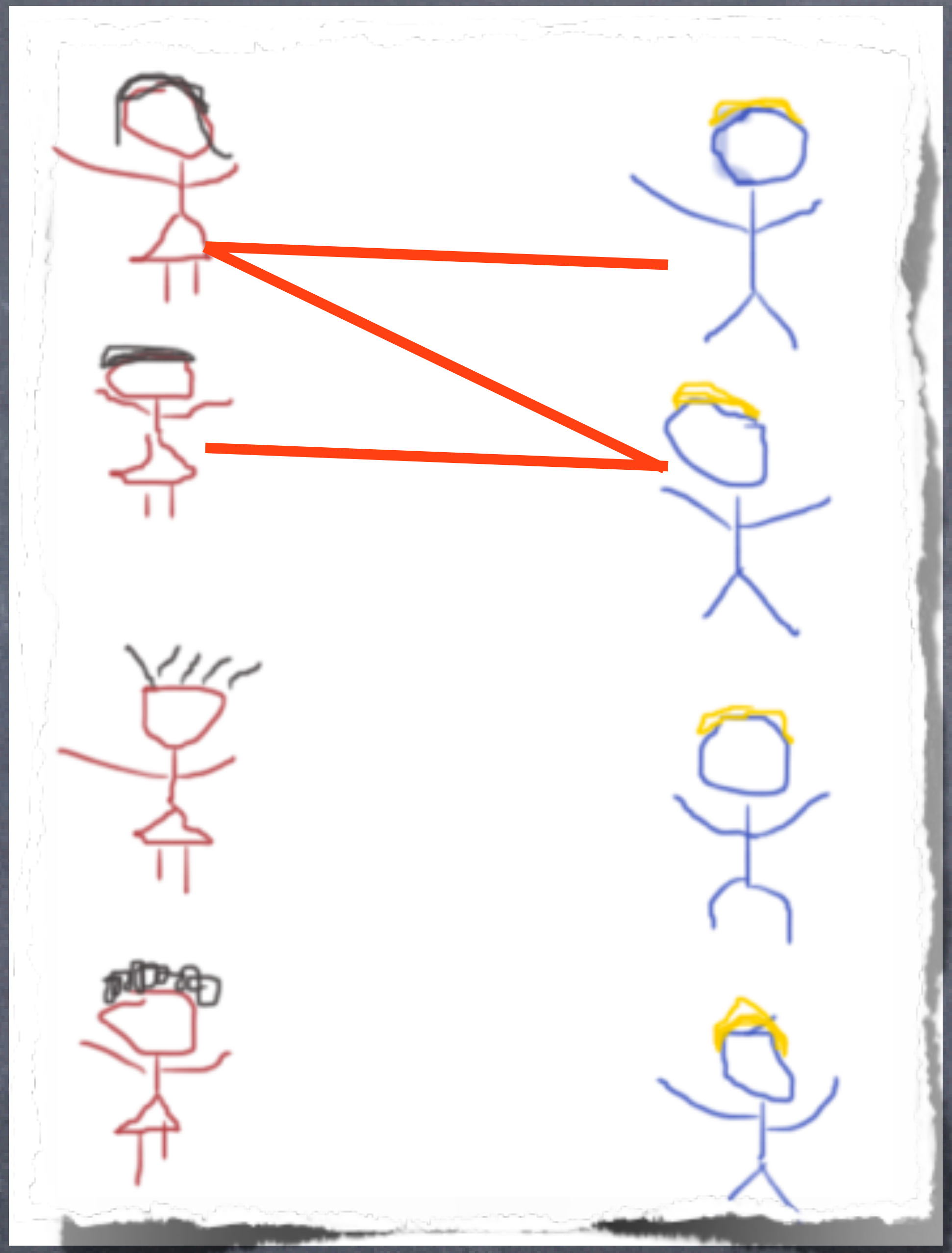
Bipartite Matchings

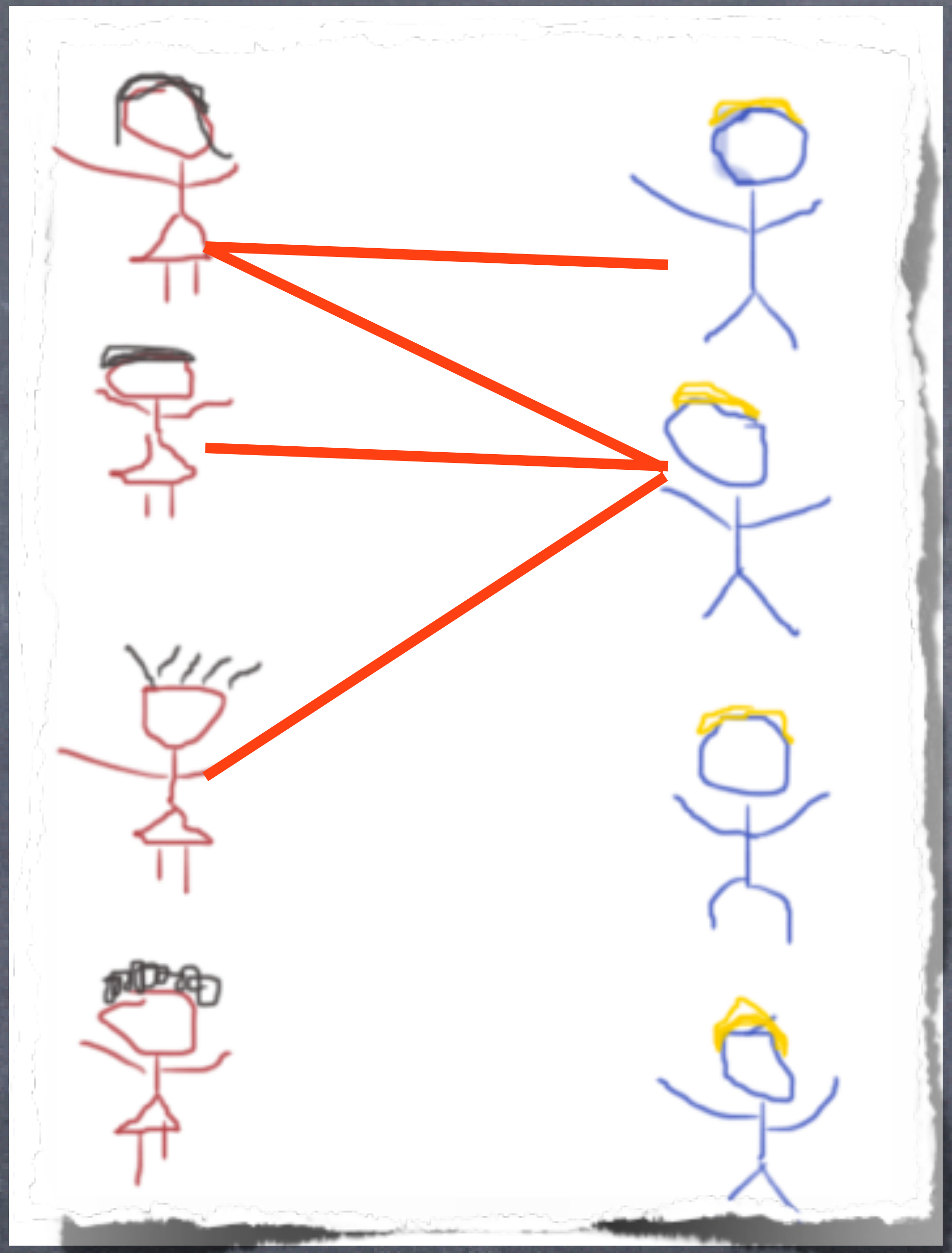
Andreas Klappenecker

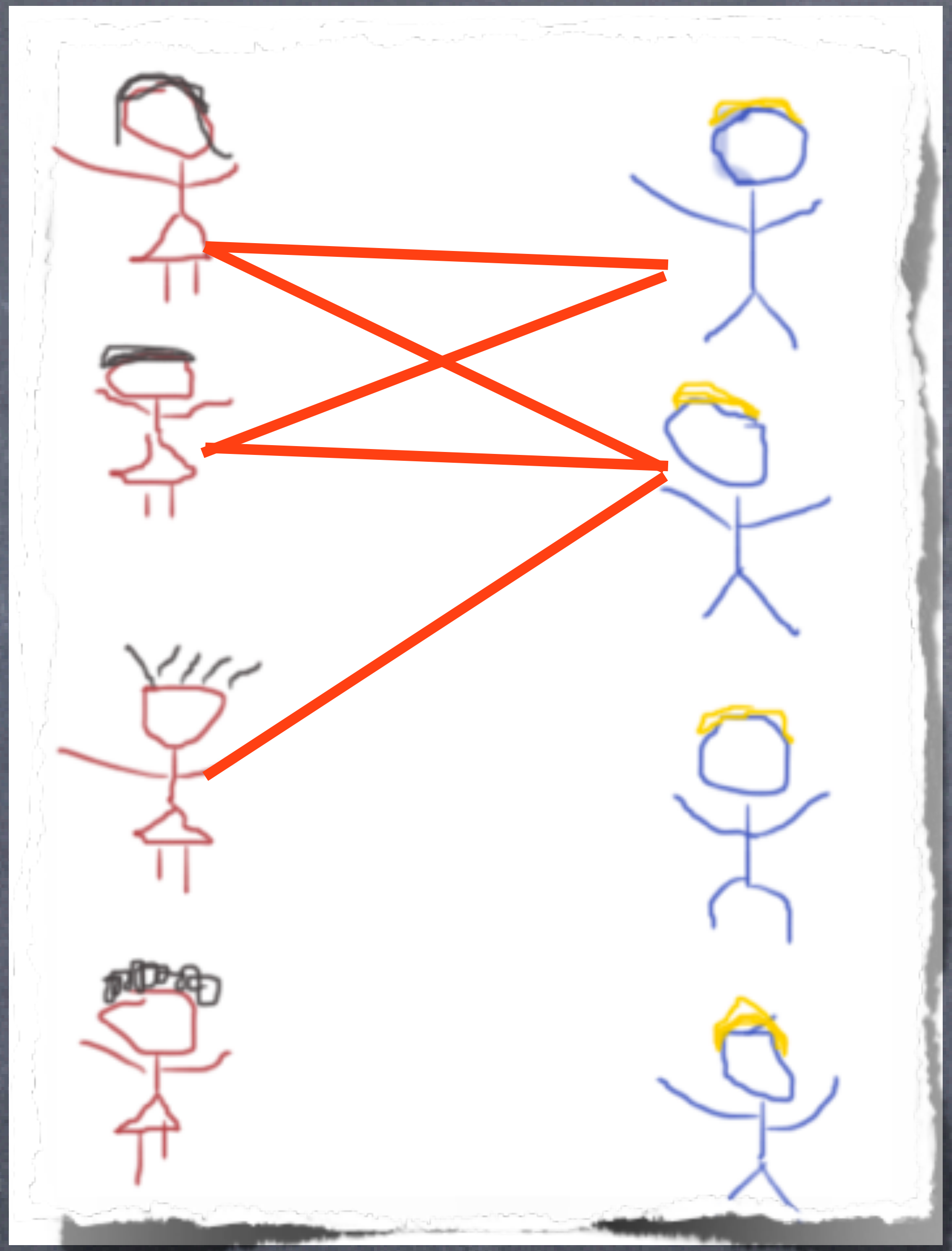


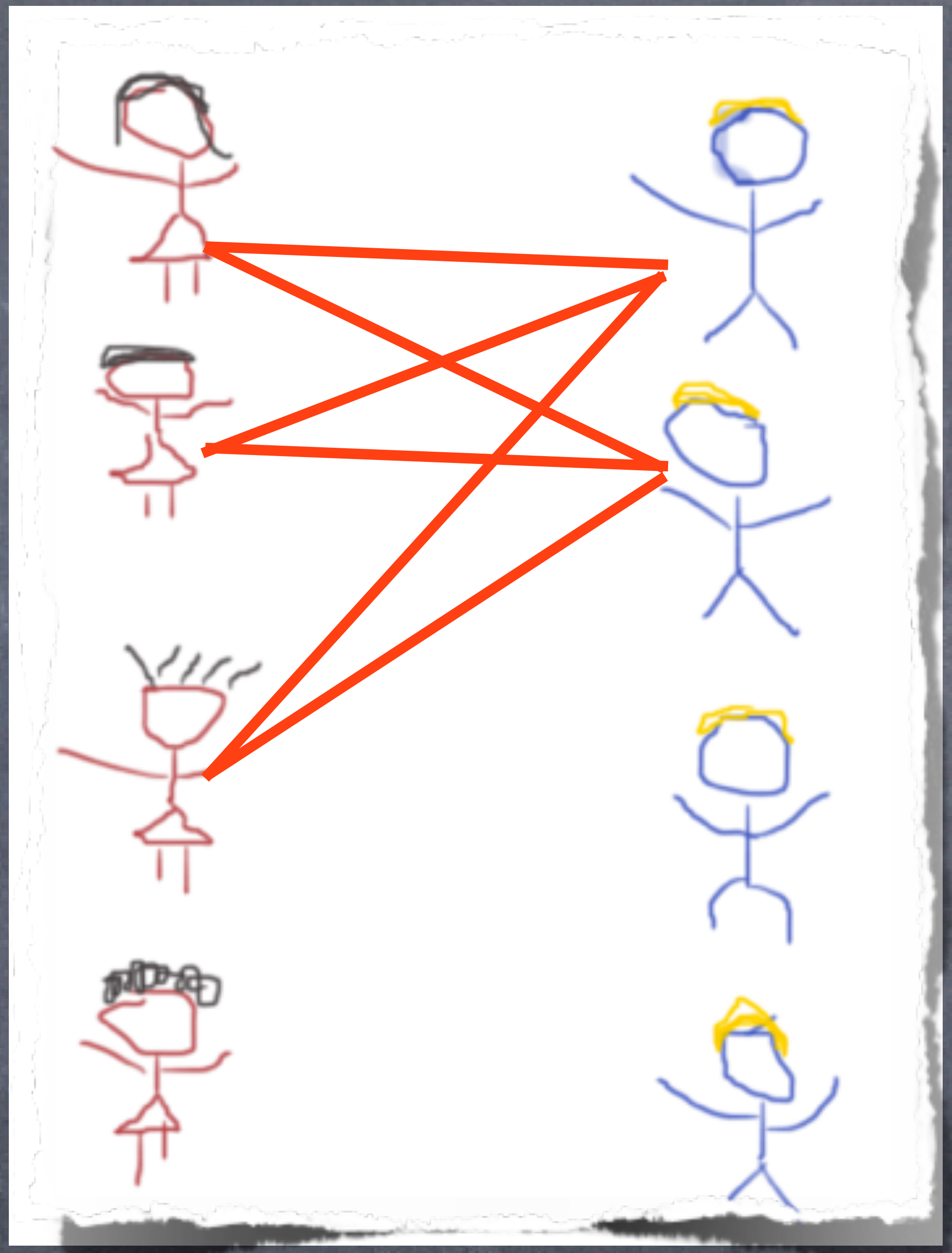


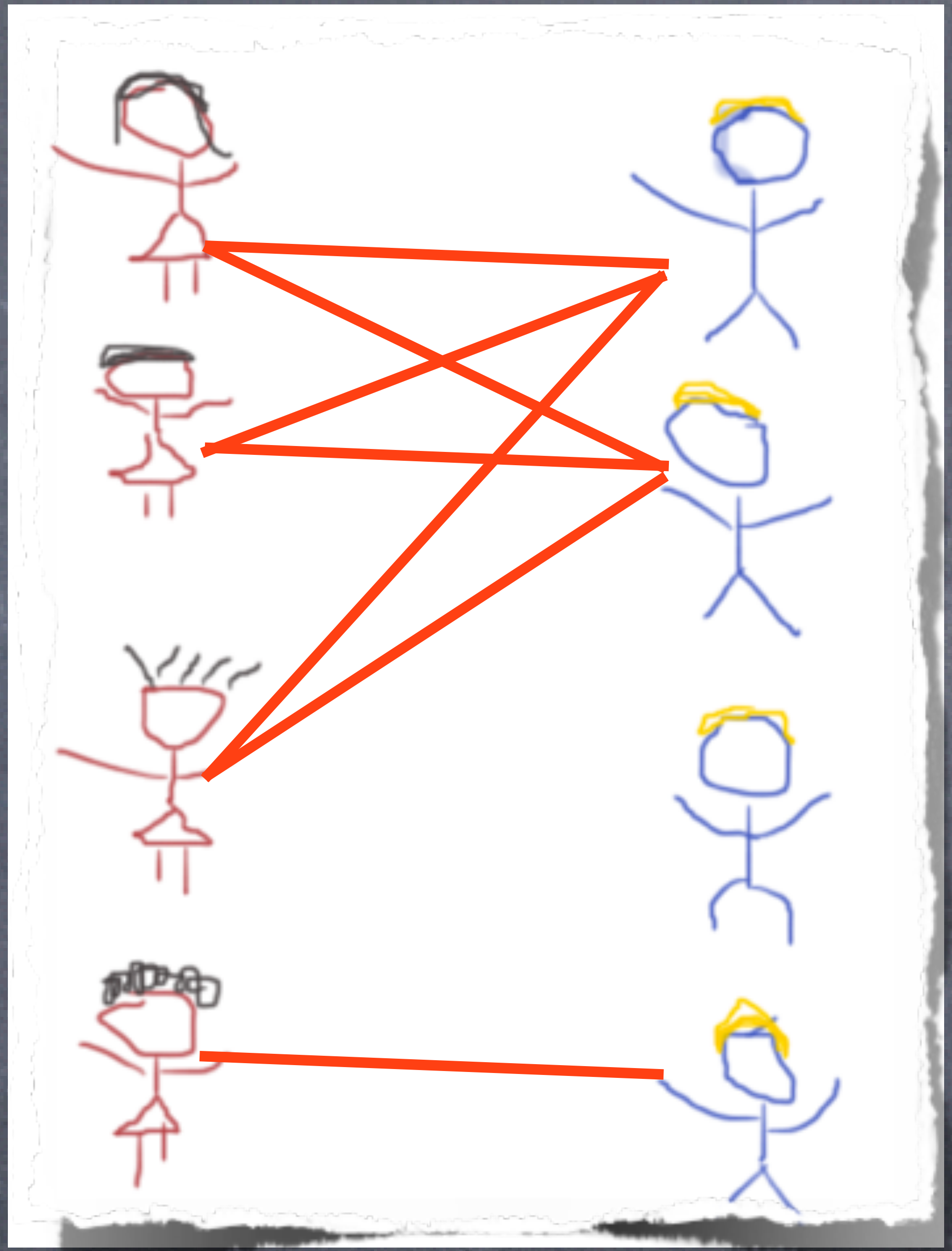


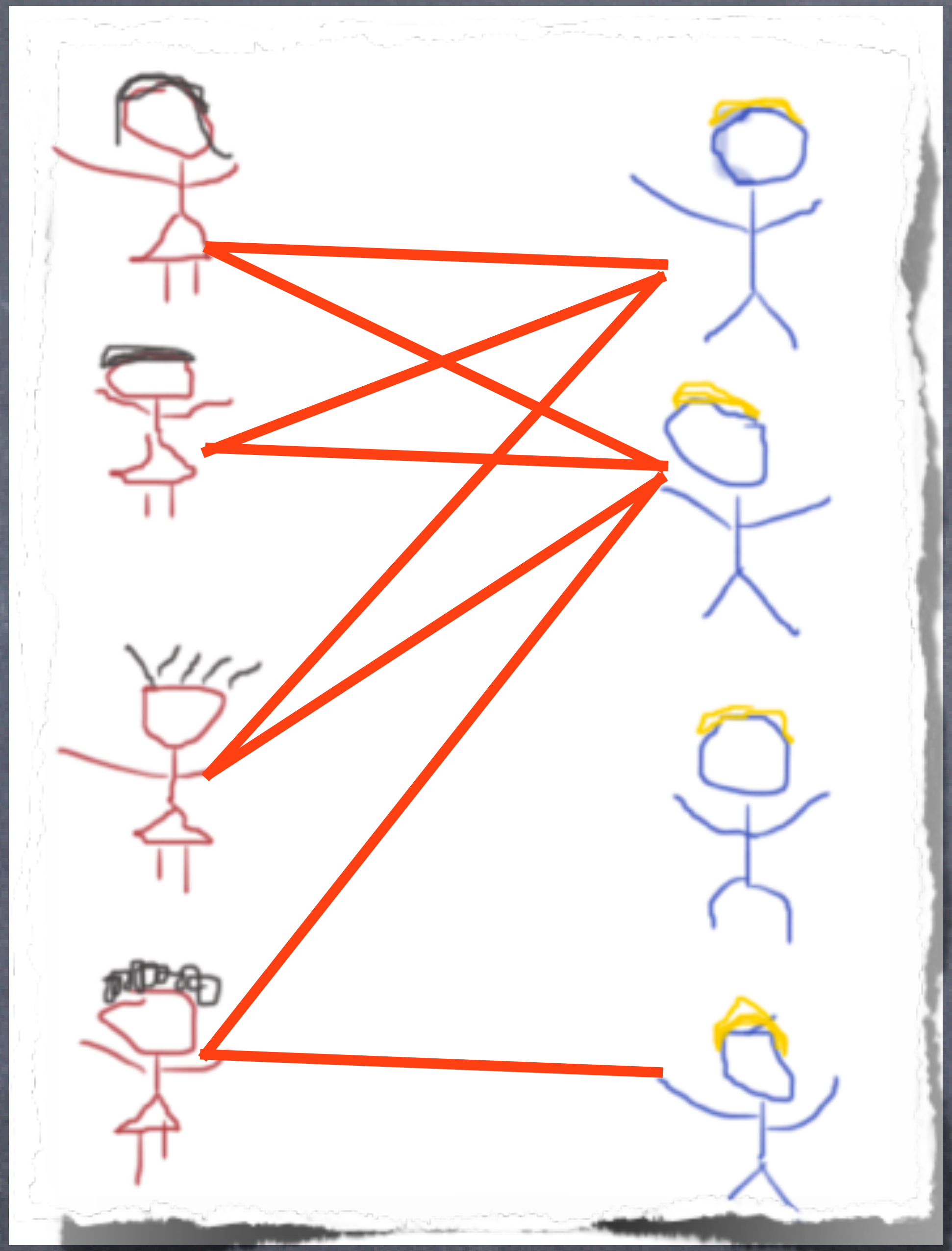


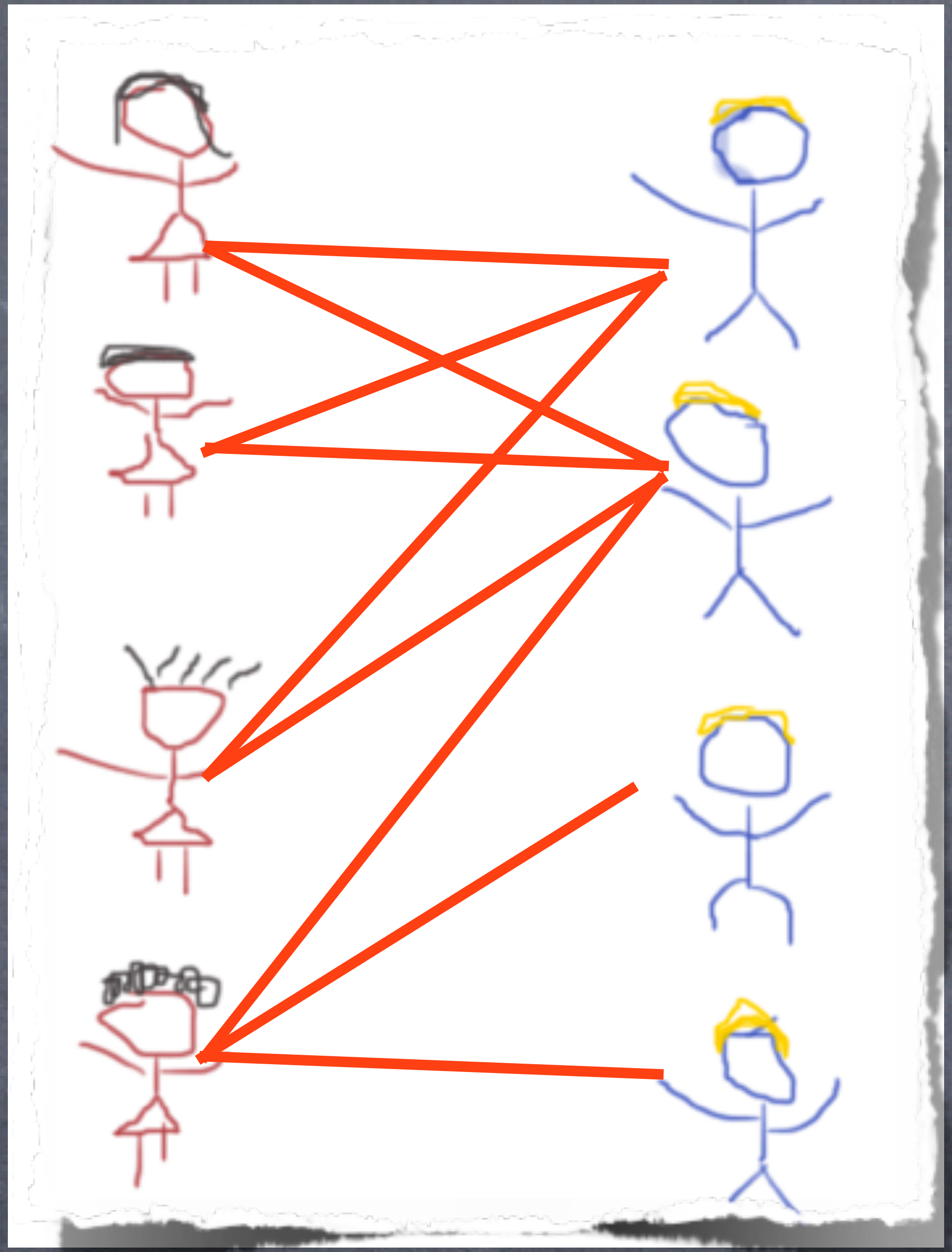


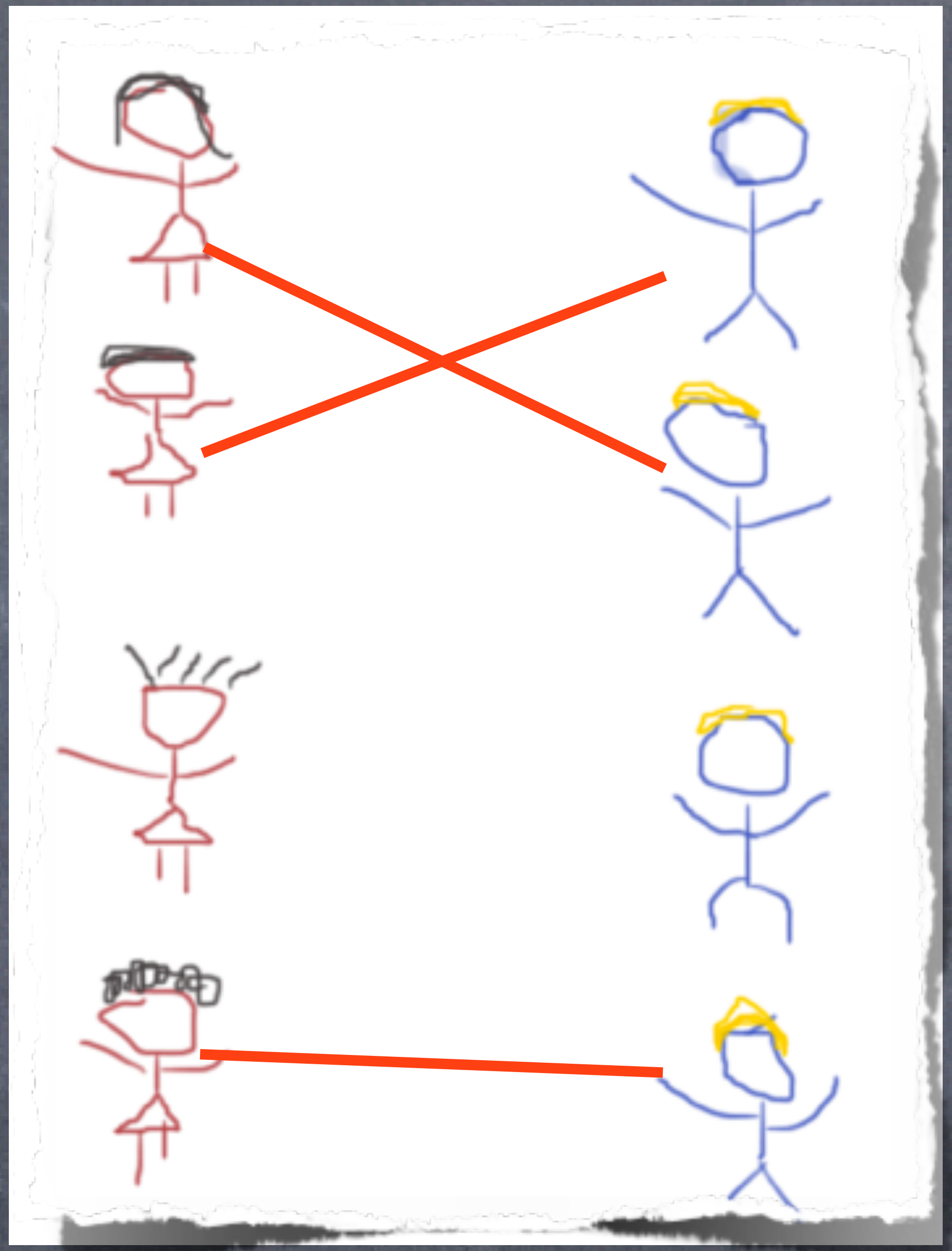






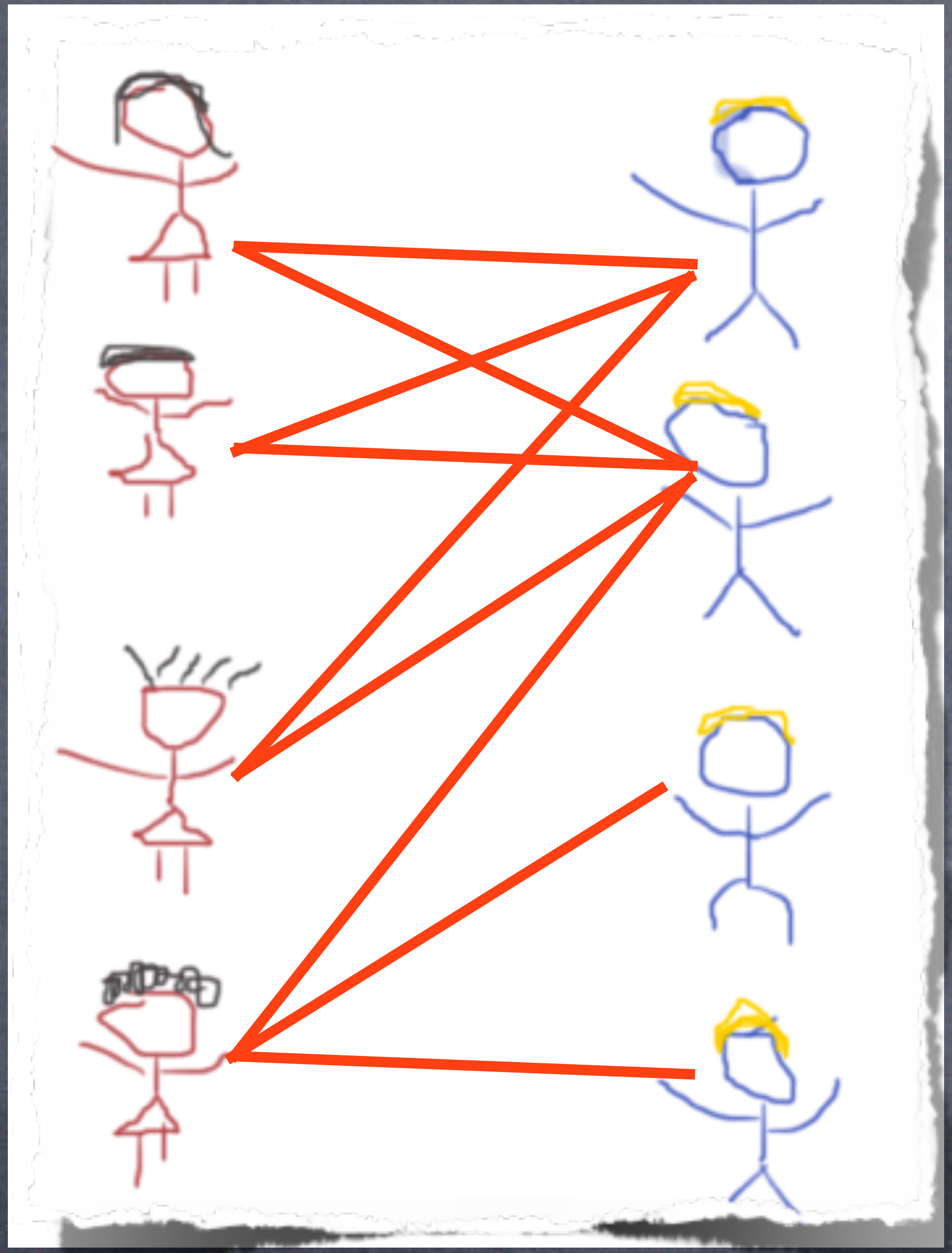






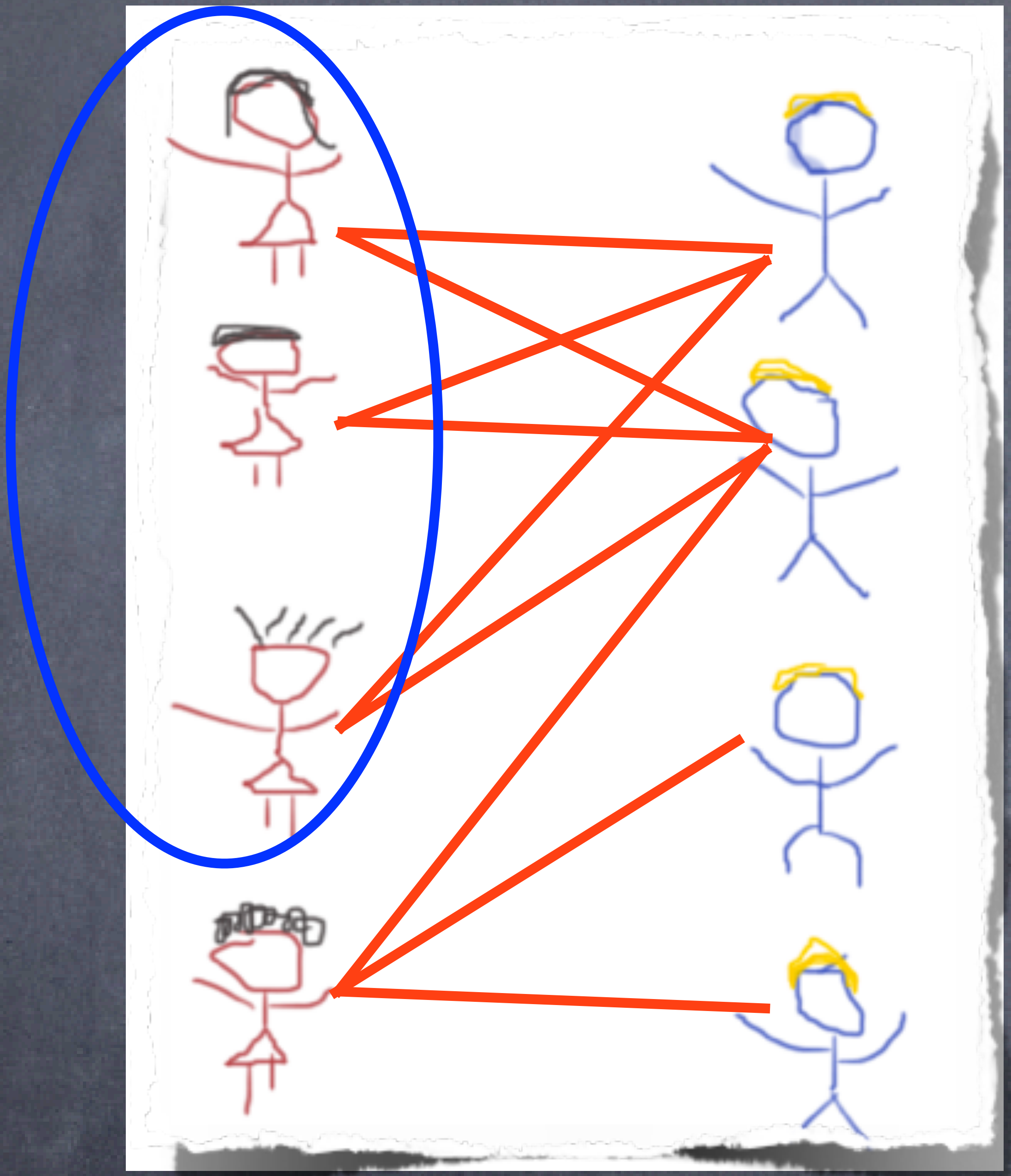
Matching Number

$m(G)$ = number of edges in a maximally large matching.



Why is $m(G) < 4$?

$m(G) = |W|$ iff $|A| <= |N(A)|$ for all $A \subseteq W$.



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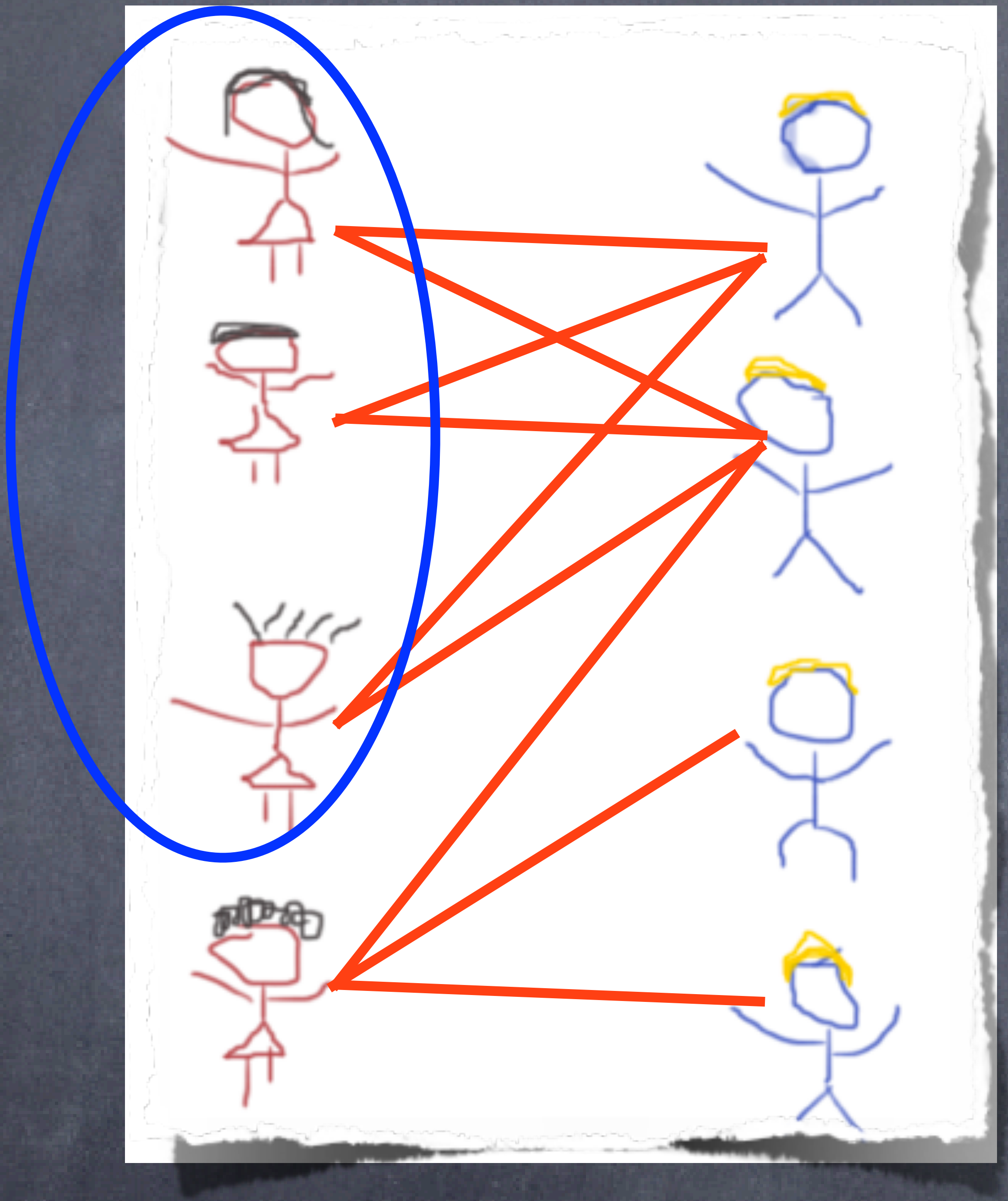

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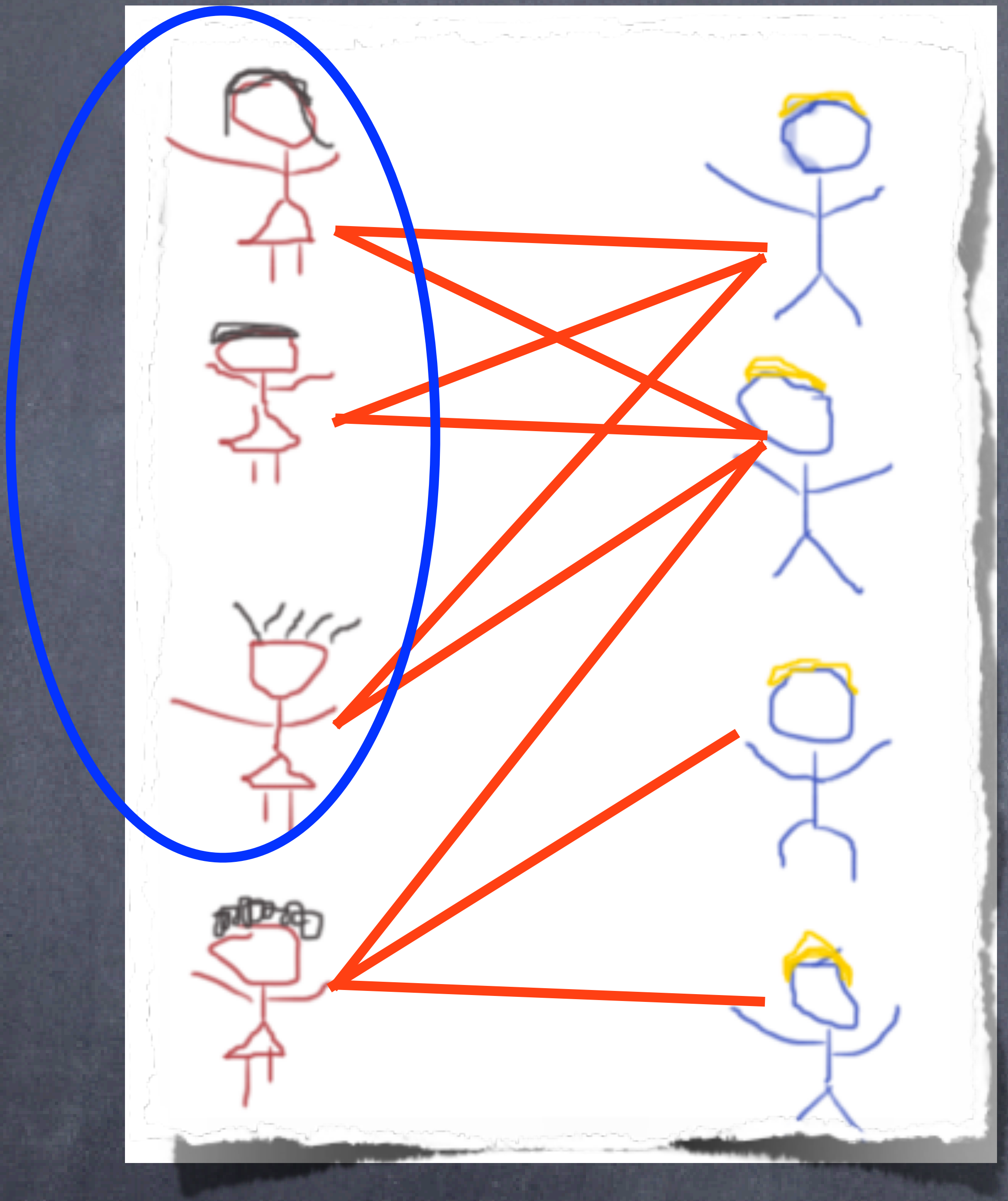

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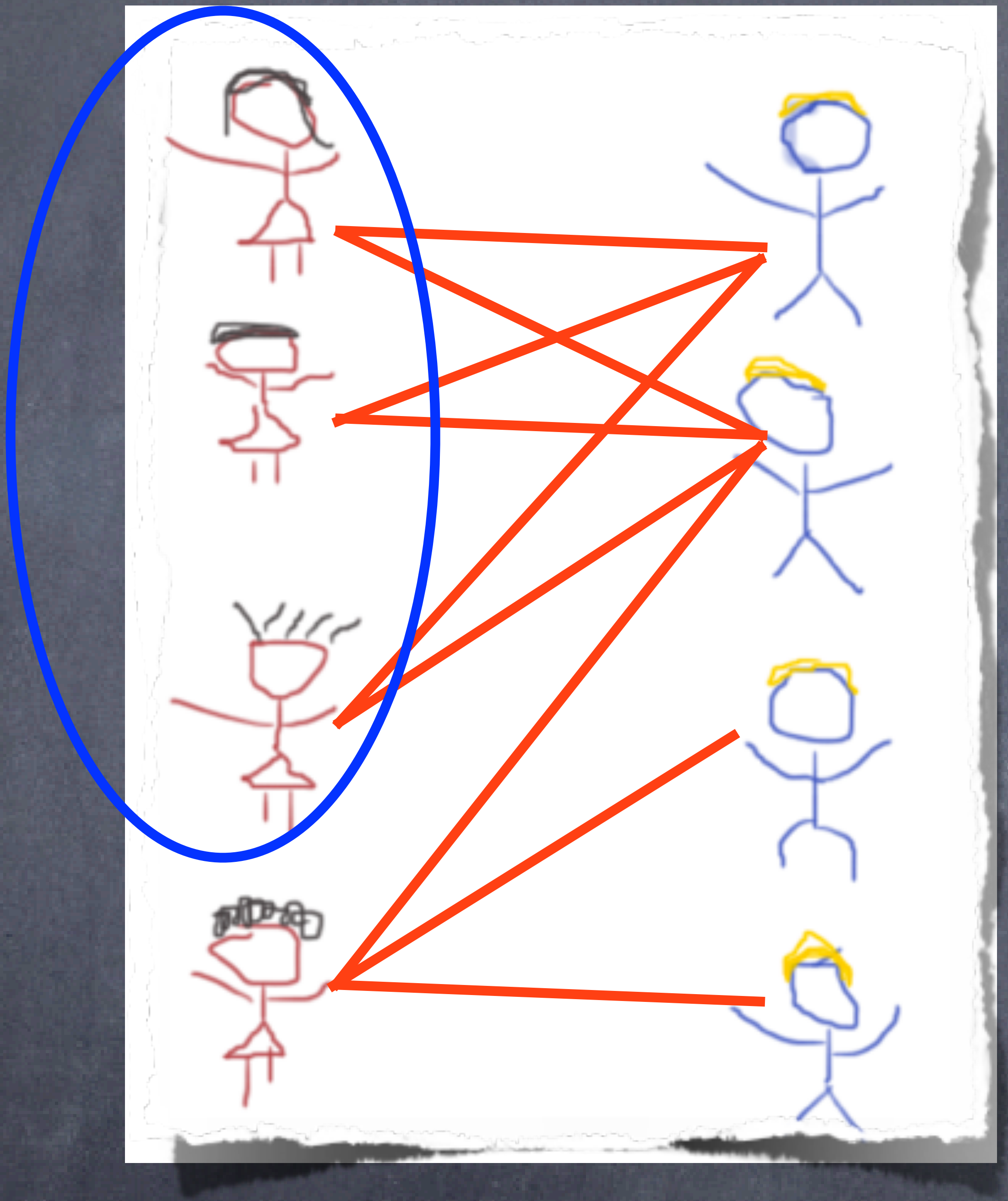


Why is m(G) < 4 ?

m(G) = |W| iff |A| <= |N(A)| for all A ⊆ W.

Marriage Theorem

Let $G = (W + X, E)$. Then

$m(G) = |W|$ iff $|A| \le |N(A)|$ for all $A \subseteq W$.

Proof: "=>" Clear.

"<=" Let $M \subseteq E$ be a matching with $|M| < |W|$. We claim that M cannot be a maximum matching.

Marriage Theorem

Let w_0 in W be unmatched in M.

Since $|N(\{w_0\})| >= |\{w_0\}|$, there exists m_1 in X such that m_1 in $N(\{w_0\})$. If m_1 is not matched in M then enlarge M by $\{w_0, m_1\}$ and stop.

Otherwise, if m_1 is matched in M with w_1, then since $|N(\{w_0, w_1\})| >= |\{w_0, w_1\}| = 2$, there exists $m_2 \neq m_1$ in $N(\{w_0, w_1\})$. If m_2 is unmatched in M, then stop.

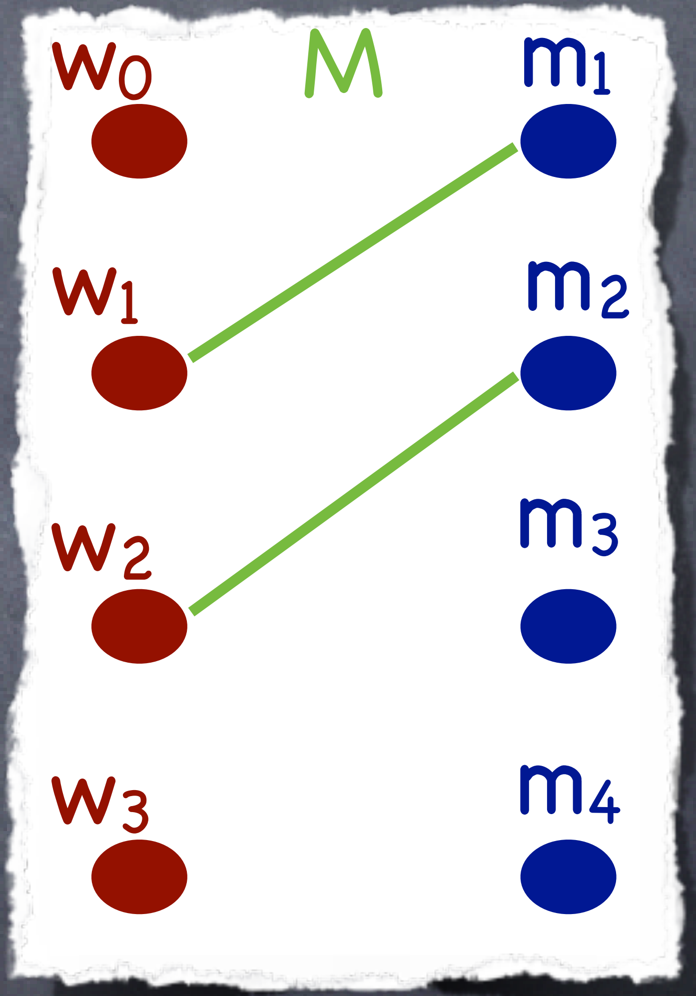
Otherwise, if m_2 matched in M with w_2 in M, then since $|N(\{w_0,w_1,w_2\})| >= \ldots$

Marriage Theorem

We proceed in the same way until we reach an unmatched m_r in M. Each m_k is neighboring to at least one w_i with i<k.

Go backward from m_r on a path P alternating between edges not in M and edges in M.

Replace edges in $P \cap M$ by edges in $P \setminus M$. Since $|P \setminus M| = |P \cap M| + 1$, we get a larger matching than A. q.e.d.



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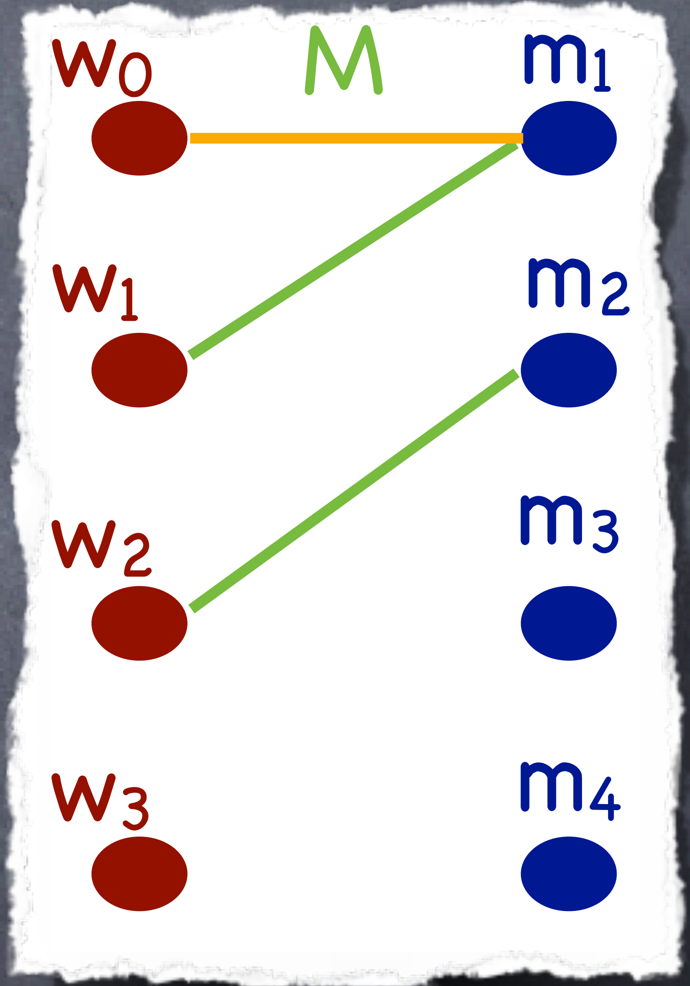

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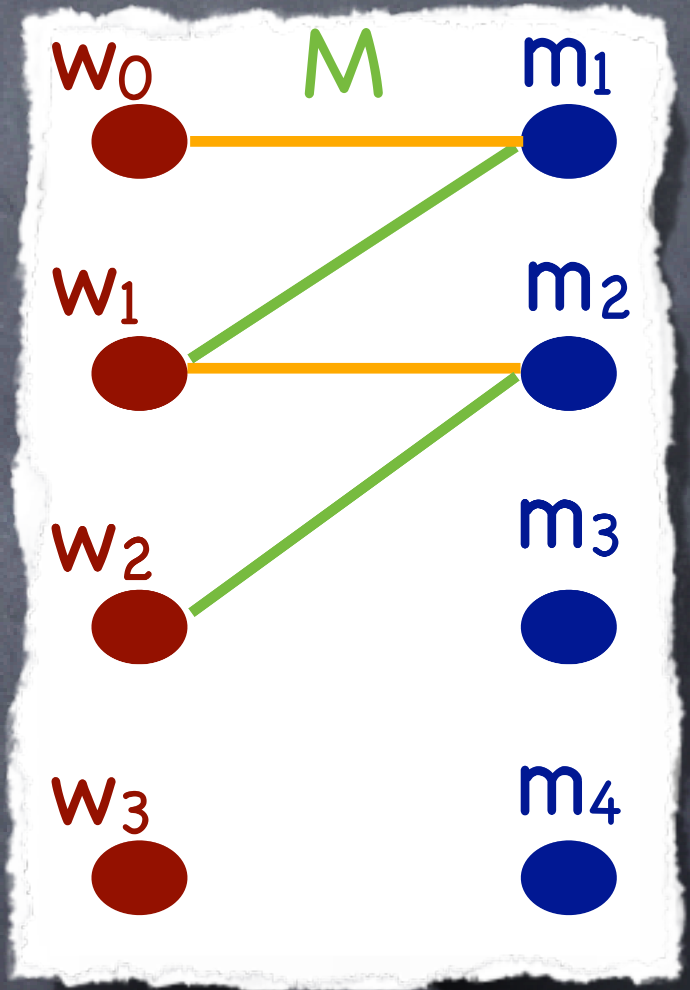

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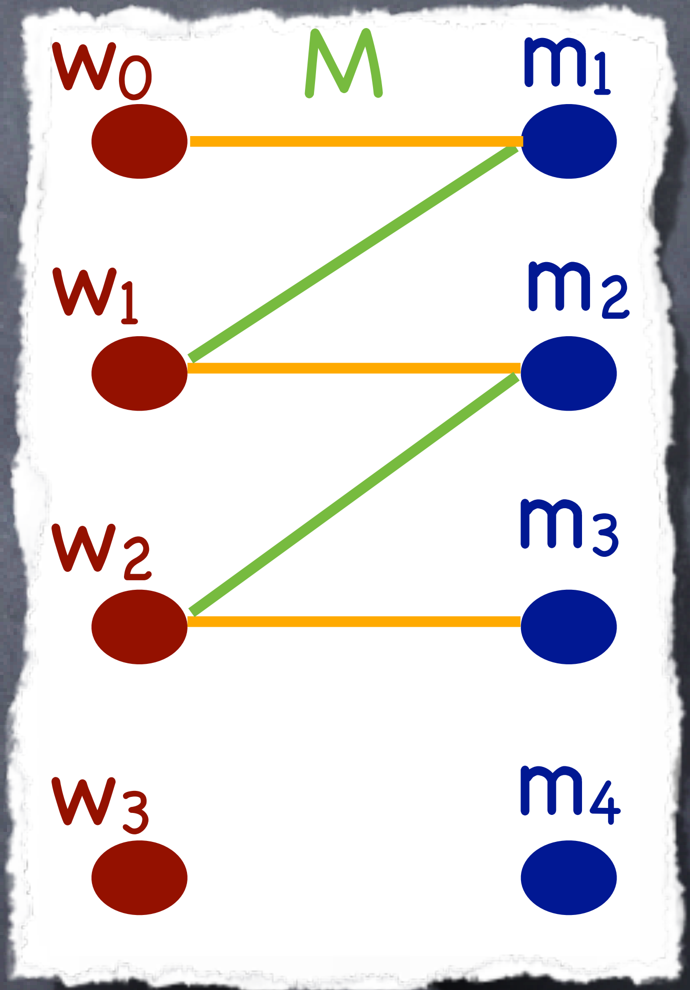


Augmenting Path

Let M be a matching in a bipartite graph G. The edges of M are called matched, the other edges in G are called unmatched.

The endpoints of edges in M are called matched, the other vertices are called free.

An M-augmenting path is a path in G such that its edges are alternating between free and matched, and its endpoints are free.

Theorem

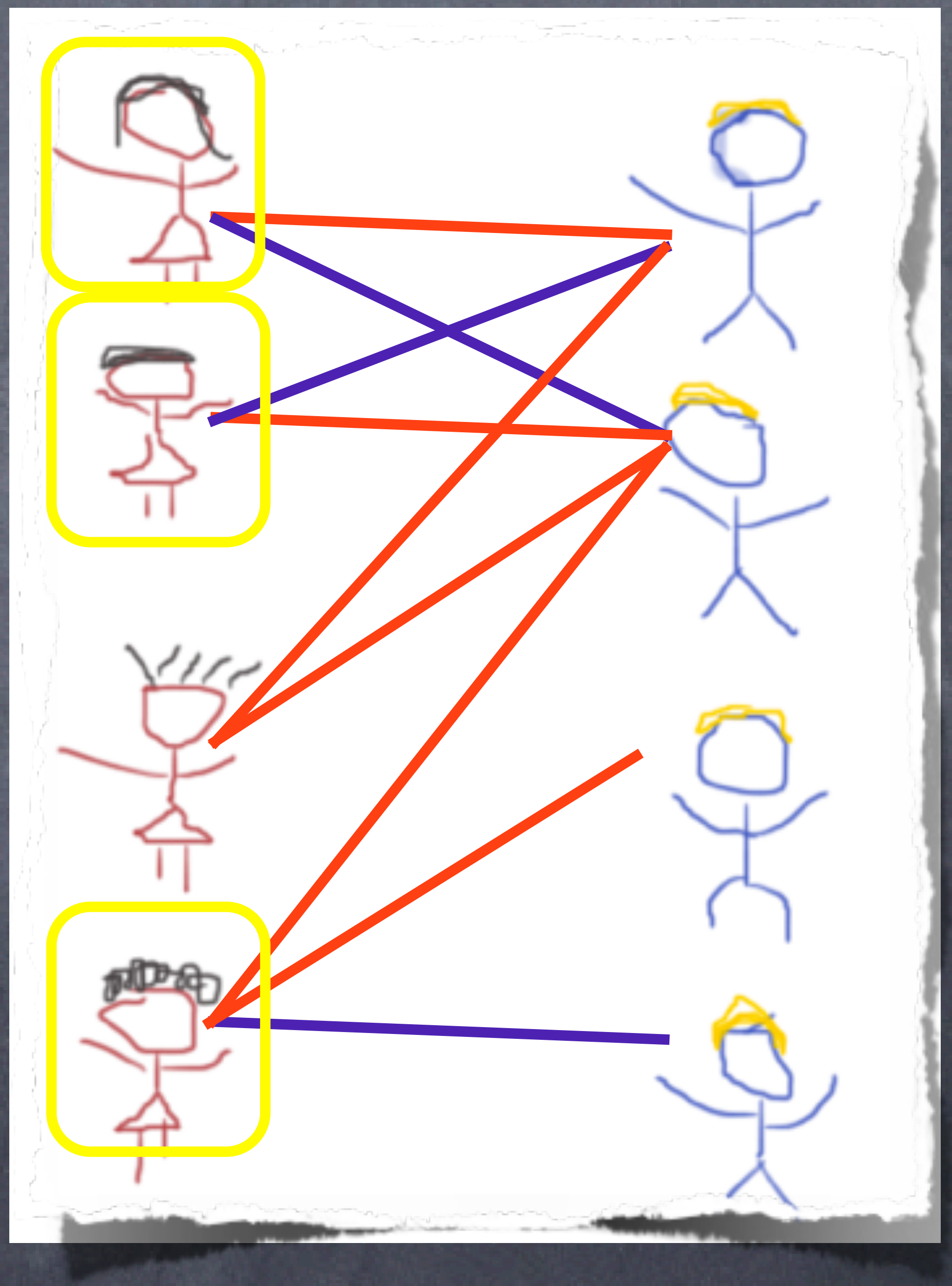
Suppose that M and M′ are matchings in G with r = |M| and s=|M′| such that s>r. Then there exist s-r vertex disjoint M-augmenting paths in G.

Transversal

Let G = (W + X, E) be a bipartite graph.

A subset U of W that can be matched in G is called a (partial) transversal of G.

The empty set is a valid transversal.



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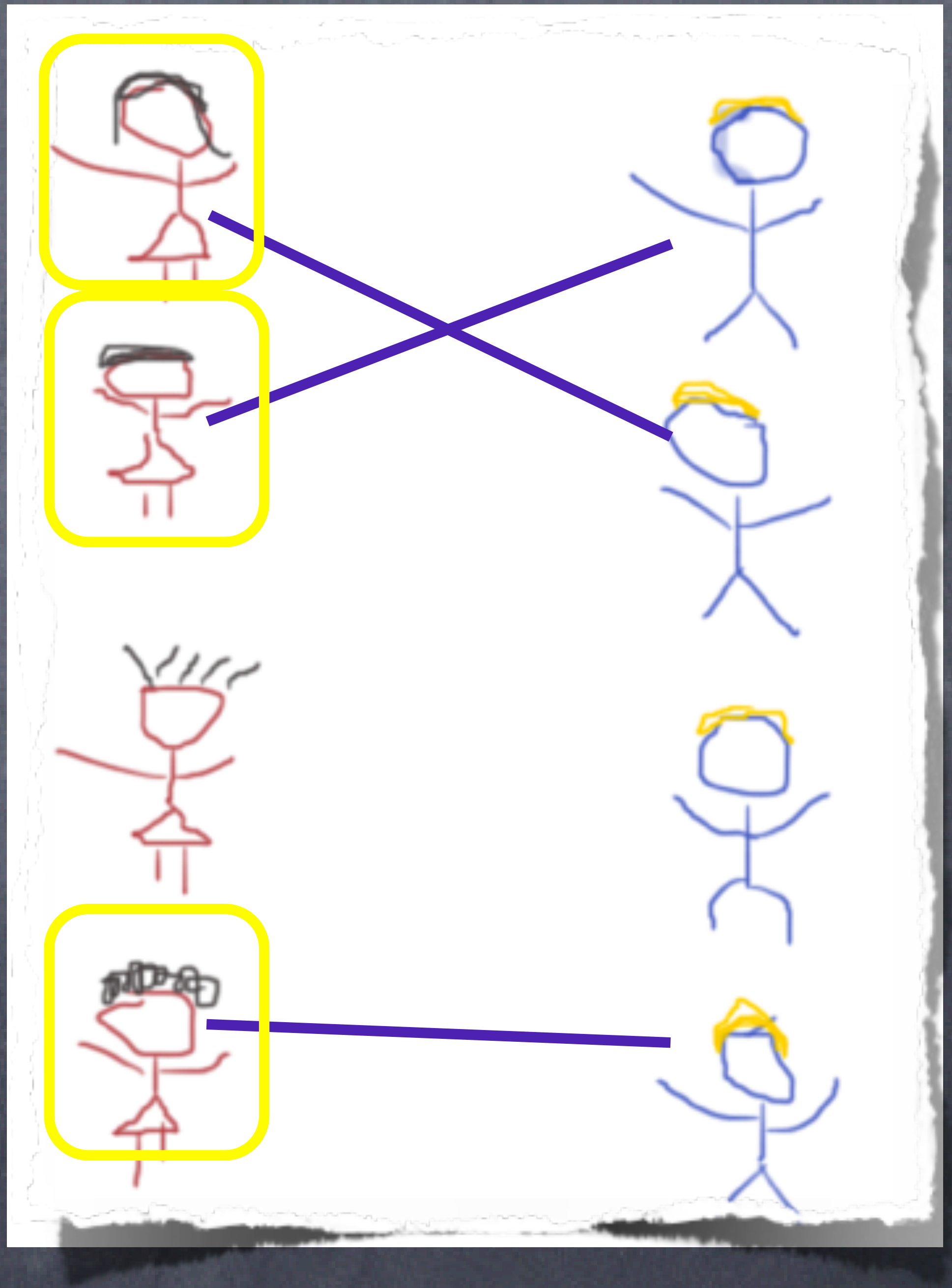


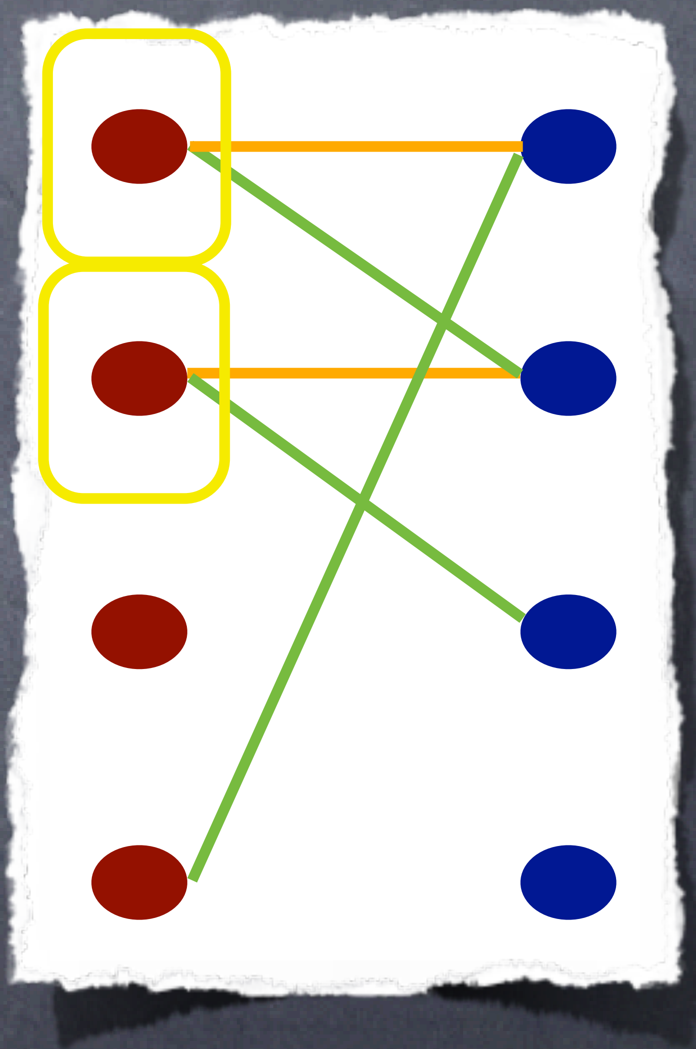
Transversal Matroid

Let G = (W + X, E) be a bipartite graph, and T⊆P(W) be the family of transversals of G. Then (W, T) is a matroid.

1) ∅ in T, so T is nonempty

2) If U in T, and V ⊆ U, then V in T

3) Exchange axiom. Consider U, V in T with |U| < |V|. Let M and M′ be the corresponding matchings, so |M|<|M′|. Form an M-augmenting path P. Swap matched edges with free edges on P to form matching with |M|+1 edges.



Transversal Matroid

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1) $\varnothing$ in T, so T is nonempty

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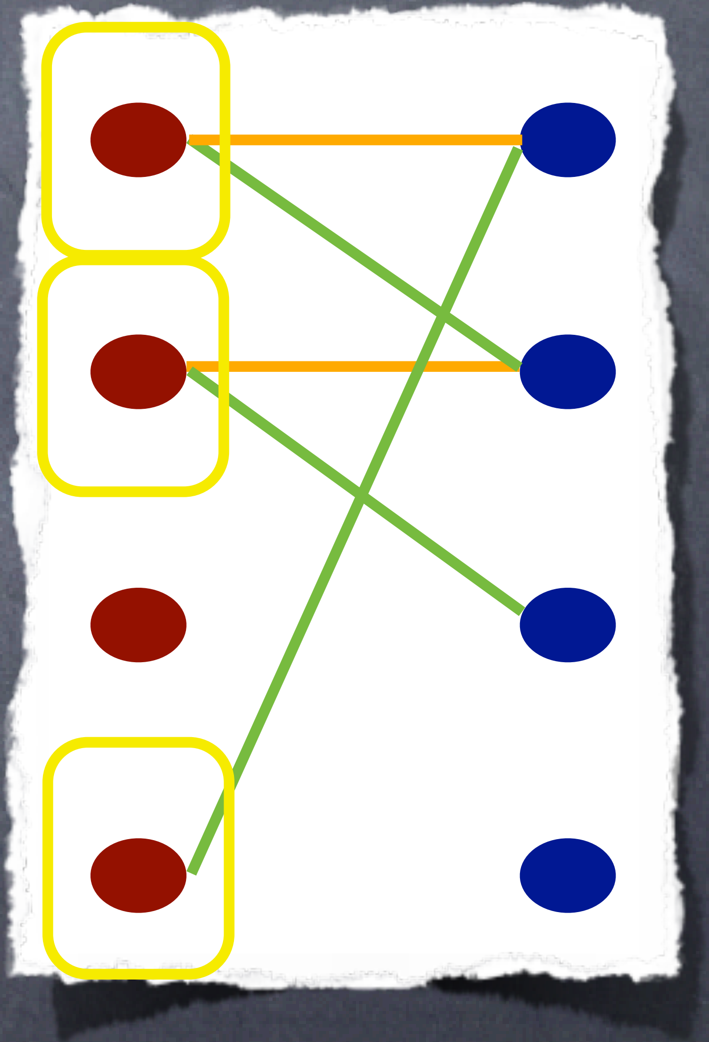

Greedy Algorithm

The generic greedy algorithm using the transversal matroid of a bipartite graph will find the maximal subset of W that has maximum weight.

Example Application

Let W = {set of wood carving jobs}, X = {set of CNC woodcarving routers}, w(j) = profit when job j is done.

The graph indicates which jobs can be performed on which CNC router.

The greedy algorithm will return the set of jobs that can be performed that will give the maximal profit.

Hopcroft-Karp Algorithm

A refinement of the generic greedy algorithms for transversal matroids leads to the Hopcroft-Karp algorithm for bipartite matching.

The worst case running time is $O(m\ n^{1/2})$ for bipartite graphs with m edges and n vertices.